MAT122 · Lectures

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Outline

Today's lesson will cover definition of function and average rate of change.

- §1.3 and 1.5

§1.3 Definition of a Function

A function is relation that assigns any object from the domain of a function to exactly one object in the codomain of a function. Let f

be a function and A and B are sets.

- f is a function on A if domain of f is A.
- f is a function into B if codomain of f is B.
- f is a function onto B if codomain of f is B.
- f is one-to-one means that f attains different values for different elements from its domain.

§1.3 Graph of a function

In mathematics, the graph of a function f is the collection of all ordered pairs $(x, f(x))$. In particular, if x is a real number, graph means the graphical representation of this collection, in the form of a curve on a Cartesian plane, together with Cartesian axes, etc.

To determine whether a relation is a function, if any vertical line drawn through the graph interests no more than once.

§1.5 Average rate of change.

The average rate of change of a function $f(x)$ over an interval between two points $(a, f(a))$ and $(b, f(b))$ is the slope of the secant line connecting the two points:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(b) - f(a)}{b - a}.$$

§1.5 Average rate of change con't

For example, to calculate the average rate of change between the points: $(0, -2) = (0, f(0))$ and $(3, 28) = (3, f(3))$

where $f(x) = 3x^2 + x - 2$ we would:

$$\frac{f(3) - f(0)}{3 - 0} = \frac{28 - (-2)}{3} = 10.$$

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Outline

Today's lesson will cover real numbers, inequalities, linear equations, and exponents.

- §1.1 and 1.2

§1.1 Real numbers

The set $\mathbb{R}$, real numbers, includes all rational numbers (fractions), all algebraic numbers, and transcendental numbers—π, e , and more. The real numbers can be drawn as the real number line. The real numbers is an ordered set. So order on the real numbers is expressed by inequalities with $a < b$ meaning to "a is to the left of b", and $a > b$ meaning "a is the right of b".

(i) $a < b$ means a is less than b.

(ii) $a > b$ means a is greater than b.

(iii) $a \leq b$ means a is less than or equal to b.

(iv) $a \geq b$ means a is greater than or equal to b.

§1.1 Intervals

An interval is a set of real numbers with the property that any number that lies between two numbers in the set is also included in the set. So the set $\{x \mid a < x < b\}$ is an interval and it is denoted (a, b). The interval (a, b) is called an open interval. The interval $[a, b]$ is called a closed interval, and in set notation it is the set $\{x \mid a \leq x \leq b\}$. The intervals $[a, b)$ and $(a, b]$ are half-open or semi-open intervals. The intervals $[a, \infty), (a, \infty), (-\infty, a]$, and $(-\infty, a)$ are infinite intervals.

§1.1 Lines and slope

The slope of a line through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}, \quad \text{where } x_1 \neq x_2.$$

The Slope-Intercept form of a line is expressed by

$$y = mx + b,$$

where m is slope and b is y-intercept.

The Point Slope form of a line is

$$y - y_1 = m(x - x_1)$$

§1.2 Exponents

Numbers may be expressed with exponents, as in $2^3 = 2 \cdot 2 \cdot 2 = 8$. More generally, for any positive integer n, x^n means the product of n x's.

$$x^n = \overbrace{x \cdot x \cdots x}^{n}$$

§1.2 Properties of Exponents

(i) $x^m \cdot x^n = x^{m+n}$

(ii) $x^m / x^n = x^{m-n}$

(iii) $(x^m)^n = x^{m \cdot n}$

(iv) $(xy)^n = x^n y^n$

(v) $(x/y)^n = x^n / y^n$

§3.1 Exponential functions

An exponential function can be defined by $f(x) = a^x$, where a is base and x is the exponent.

Example (1)

$$f(x) = 10^x,$$

where $b = 10$.

Example (2)

$$g(x) = \left(\frac{1}{2}\right)^x,$$

where $b = \frac{1}{2}$.

§3.1 Useful Definitions

Definition: If an exponential function is defined by $f(x) = a^x$ and $f > 1$, then a is the growth factor. If f is increasing, then $a > 1$. If f is decreasing, then $a < 1$. In this case, a is the decay factor.

Definition: A horizontal axis having equation $y = 0$ is called a horizontal asymptote of the graph of a function defined by $f(x) = a^x$, where $a > 0$ and $a \neq 1$. The graph of the function gets closer and closer to the x-axis $(y = 0)$ as the input gets farther from the origin, in the negative direction.

§3.3 Population Growth

Exponential functions represent quantities that change by a constant growth or decay factor.

The growth factor,b, is determined from the growth rate, r, by writing r in decimal form and adding 1: $b = 1 + r$.

The growth rate, r, is determined from the growth factor, b, by subtracting 1 from b and writing the result in percent form.

The equation $P(t) = 541(1.032)^t$ has the general form $P(t) = P_0(1 + r)^t$, where r is the annual growth rate, $(1 + r)$ is the growth factor or the base, b, of the exponential function, t is the time in years, and P_0 is the initial value, the population when $t = 0$.

§3.3 Population Growth

The equation $C(n) = 18(0.80)^n$ has the general form $C = C_0(1 - r)^n$, where r is the decay rate, $(1 - r)$ is the decay factor the base of the exponential function, n is the number of treatments, and C_0 is the initial value, the concentration when $n = 0$.

§3.5 Compound Interest

If interest is compounded, then the current balance is given by the formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt},$$

where A is the current amount, or balance, in the account;

- P is the principal (the original amount deposited);
- r is the annual interest rate (annual percentage rate in decimal form);
- n is the number of times per year that interest is compounded; and
- t is the time in years the money has been invested.

§3.5 Bond Price

$$
\begin{aligned}
P_c &= \sum_{t=1}^{m} \frac{C}{u^t} + \frac{F}{(1+r)^T}, \\
&= \sum_{t=1}^{m} \frac{C}{(1+\frac{r}{n})^t} + \frac{F}{(1+\frac{r}{n})^{nT}},
\end{aligned}
$$

Cash flows

- The periodic coupon payments C, each of which is made n times (n is usually 2) every year
- The par or face value F, which is payable at maturity of the bond after T years.(NB final year payments will include the par value plus the coupon payments for the year)

Discount rate: the required (annually compounded) yield or rate of return r

- r is the market interest rate for bonds with similar terms and risk ratings
- m is the number of coupons to be paid over the remaining lifetime of the bond, i.e. n times T. (It is assumed that the previous coupon has just been paid.)
- u is $(1 + r/n)$ ie an interest accumulation factor over one coupon period

§3.5 Example

Example 1: Calculate the price of a bond with a par value of $\$1,000$ to be paid in ten years, a coupon rate of 10%, and a required yield of 12%. In our example we'll assume that coupon payments are made semi-annually to bond holders and that the next coupon payment is expected in six months. Here are the steps we have to take to calculate the price:

§3.11 Exponential and Logarithm Functions

If $A > 0, B > 0$, then $\log_b(A \cdot B) = \log_b A + \log_b B$, where $b > 0, b \neq 1$. Expressed verbally, this property states that the logarithm of a product is the sum of individual logarithms.

If $A > 0, B > 0$, then $\log_b\left(\frac{A}{B}\right) = \log_b A - \log_b B$, where $b > 0, b \neq 1$. Expressed verbally, this property states that the logarithm of a quotient is the difference of the logarithm of the numerator and the logarithm of the denominator.

§3.11 Exponential and Logarithmic Functions

If $A > 0$ and p is any real number, then $\log_b A^p = p \cdot \log_b A$, where $b > 0, b \neq 1$. In words, the property states that the logarithm of a power is equivalent to the exponential times the logarithm of the base.

$$\begin{aligned} \log_b x &= \frac{\log x}{\log b} \\ &= \frac{\ln x}{\ln b} \end{aligned}$$

where $a > 0 \neq 1$.

§4.2 Quadratic and Higher-Order Polynomials Functions

An equivalent form of the equation $y = f(x) = ax^2 + bx + c$ with vertex (h, k) is

$$f(x) = a(x - h)^2 + k.$$

The vertex of the parabola is (h, k), where $h = -\frac{b}{2a}$ and $k = f(h)$.

§4.2 Quadratic and Higher-Order Polynomials Functions

The vertex or turning point of a parabola equation

$$f(x) = ax^2 + bx + c$$

has the following coordinates

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right),$$

where a is the coefficient of the x^2 term and b is the coefficient of the x term.

§4.2 Quadratic and Higher-Order Polynomial Functions

Def: The axis of symmetry of a parabola is a vertical line that divides the parabola into two symmetrical parts that are mirror images in the line.

Because the vertex (turning point) of a parabola lies on the axis of symmetry, the equation of the axis of symmetry is

$$x = -\frac{b}{2a}$$

The y-intercept of the parabola defined by $f(x) = ax^2 + bx + c$ is $(0, c)$.

§4.2 Quadratic and Higher-Order Polynomials Functions

The x-intercept is the point or point (if any) where the parabola crosses the x-axis (that is, where it y-coordinate is zero).

If a parabola opens upward and the vertex is above the x-axis, there are no x-intercepts. If a parabola opens downward and the vertex is below the x-axis, there are no x-intercepts.

The domain of the general quadratic function is the set of all real numbers. If the parabola opens upward, the range is all real numbers greater than or equal to the output value of the vertex. If the parabola opens downward, the range is all real numbers less than or equal to the output value of the vertex.

§4.4 Factoring

If a and b are any tow numbers and $a \cdot b = 0$, then either a or b, or both, must be equal to zero.

Removing a Common Factor from a Polynomial: First, identity the common factor, and then apply the distributive property in reverse.

§4.4 Factoring

Factoring Trinomials by Trial and Error:

1. Remove the greatest common factor, GCF.
2. To factor the resulting trinomial into products of two binomials, try combination of factors for the first and last terms in two binomials
3. Check the outer and inner products to match the middle term of the original trinomial.
   a. If the constant term, c, is positive, both its factors are positive or both are negative.
   b. If the constant term is negative, one factor is positive and one is negative.
4. If the check fails, repeat steps 2 and 3.

§5.1 Rational Functions

In mathematics, a rational function is any function which can be written as the ratio of two polynomial functions. In the case of one variable, x, a rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomial function in x and Q is not the zero polynomial. The domain of f is the set of all points x for which the denominator $Q(x)$ is not zero.

§5.1 Rational functions

An irrational function is a function that is not rational. That is, it cannot be expressed as a ratio of two polynomials.

Examples:

The rational function

$$f(x) = \frac{x^3 - 2x}{2(x^2 - 5)}$$

is not defined at $x = \pm\sqrt{5}$.

§5.1 Rational functions

The rational function

$$f(x) = \frac{x^2 + 2}{x^2 + 1}$$

is defined for all real numbers, but not for all complex numbers.

Let $f(x)$ be a rational function, which is defined below:

$$f(x) = \frac{x^2 + x - 20}{x^2 - 3x - 18}$$

§5.1 Rational functions

To understand the behavior of a rational function it is very useful to see its polynomials in factored form. The polynomials in the numerator and the denominator of the above function would factor like this:

$$f(x) = \frac{(x+5)(x-4)}{(x+3)(x-6)}$$

Now the roots of the denominator are obviously $x = -3$ and $x = 6$. That is, if x takes on either of these two values, the denominator becomes equal to zero. Since one can not divide by zero, the function is not defined for these two values of x. We say that the function is discontinuous at $x = -3$ and $x = 6$.

§5.1 Rational functions

Other values for x do not cause the function to be undefined, so, we say that the function is continuous at all other values for x. In other words, all real numbers except -3 and 6 are allowed as inputs to this function. The domain for the function in interval notation is

$$(-\infty, -3) \cup (-3, 6) \cup (6, +\infty)$$

The x-intercepts for this function would be where the output, or y-value, equals zero. So, the x-intercepts are $x = -5$ and $x = 4$.

What about the y-intercept? Well, they occur where the input, or x, value equals zero. And so, the y-intercept is $(0, 10/9)$.