MAT209 EXAM I SOLUTIONS

MAT 209 · SPRING 2009

You must show all work to get full credit.

Problem 1 (20 points)**.** Solve the differential equation

$$\frac{dy}{dx} = y^2$$

subject to the initial condition $y(0) = 1$.

Solution: It follows that

$$-\frac{1}{y} = \int \frac{\mathrm{d}y}{y^2} = \int \mathrm{d}x = x + c.$$

Multiply both sides of the equality by one so that

$$\frac{1}{y} = -x - c = -x + c.$$

The reciprocal of $1/y$ is y, and so

$$y = \frac{1}{c - x}.$$

We have found the general solution to the ODE. We can use the initial condition to solve for c in the general solution:

$$\begin{aligned} 1 &= y(0) \\ &= \frac{1}{c-0} \\ &= \frac{1}{c}. \end{aligned}$$

But then, $c = 1$. Therefore, the solution to the initial value problem is

$$y = \frac{1}{1 - x}.$$

Problem 2 (20 points)**.** Solve the differential equation

$$\frac{dy}{dx} = -xy$$

subject to the initial condition $y(0) = \frac{1}{\sqrt{2\pi}}$.

Solution: The ODE is separable since the right side is a product of the independent and dependent variables. We can use the technique of separation of variables. It follows that

$$\ln|y| = \int \frac{\mathrm{d}y}{y} = -\int \mathrm{d}x = -\frac{x^2}{2} + c.$$

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Exponentiating both sides of the equality, we get

$$|y| = e^{-\frac{x^2}{2}+c} = e^c e^{-\frac{x^2}{2}}.$$

Note that $|y| = \pm y$. But then,

$$\begin{aligned} y &= \pm e^c e^{-\frac{x^2}{2}} \\ &= Ce^{-\frac{x^2}{2}}. \end{aligned}$$

We have found the general solution to the ODE. We can use the initial condition to solve for C in the general solution:

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} &= y(0) \\ &= Ce^{-\frac{(0)^2}{2}} \\ &= Ce^0 \\ &= C. \end{aligned}$$

Therefore, the solution to the initial value problem is

$$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Problem 3 (10 points)**.** Consider the initial value problem

$$\frac{dy}{dx} = x^2 - y$$

where $y(0) = 1$. Can this ODE be solved using *Separation of Variables*? If not, What other methods that we have learned could be used to solve initial value problem?

Solution: If we try to factor the right side of the ODE, we get

$$\frac{dy}{dx} = x^2 - y = x\left(x - \frac{y}{x}\right).$$

The second factor is a function of both x and y. The ODE is not separable. If we want to know the value of y at some point $x > 0$, we can use Euler's Method or the Modified Euler Method to approximate $y(x)$ given some step size Δx.

Problem 4 (20 points)**.** Consider the initial value problem

$$\frac{dy}{dx} = x - y$$

where $y(0) = 1$. Use the Eulers method with step size 0.20 to find the approximate value of $y(1)$.

Solution: Note that $\Delta x = (b - a)/n$. Since the step size is 0.20, $a = 0$, and $b = 1$; it must be that $n = 5$. We must have four subintervals. Recall Euler's formula:

$$y(x + \Delta x) \approx y(x) + y'(x)\Delta x.$$

We will rewrite this expression as

$$y_{n+1} = y_n + y'_n \Delta x.$$

For $n = 0$, it follows that

$$y_1 = y_0 + y'_0 \Delta x,$$

where $y_0 = y(0) = 1, y_0' = x_0 - y_0, \Delta x = 0.20, x_0 = 0$. So

$$\begin{aligned} y_1 &= 1 + (0-1)(0.20) \\ &= 0.8000. \end{aligned}$$

For $n = 1$, we have

$$\begin{aligned} y_2 &= y_1 + y_1' \Delta x \\ &= y_1 + (x_1 - y_1)\Delta x \\ &= (0.8000) + ((0.2000 - 0.8000)(0.20) \\ &= 0.6800 \end{aligned}$$

For $n = 2$, we get

$$\begin{aligned} y_3 &= y_2 + y_2' \Delta x \\ &= y_2 + (x_2 - y_2)\Delta x \\ &= (0.6800) + (0.4000 - 0.6800)(0.20) \\ &= 0.6240. \end{aligned}$$

For $n = 3$, we have

$$\begin{aligned} y_4 &= y_3 + y_3' \Delta x \\ &= y_2 + (x_2 - y_2)\Delta x \\ &= (0.6240) + (0.6000 - 0.6240)(0.20) \\ &= 0.6192. \end{aligned}$$

For $n = 4$, we have

$$\begin{aligned} y_5 &= y_4 + y_4' \Delta x \\ &= y_4 + (x_4 - y_4)\Delta x \\ &= (0.6192) + (0.8000 - 0.6192)(0.20) \\ &= 0.6554. \end{aligned}$$

Recall that $y_5 = y(1)$. Therefore, $y(1) = 0.655$.

Problem 5 (40 points)**.** Use geometric analysis to analyze the differential equation

$$\frac{dy}{dt} = (y-1)(y-3)$$

Your answer should include:

a) a graph of $g(y)$ vs. y , where $g(y)$ is the derivative of y as a function of y.
b) sketches of the solution curves $y(t)$ for the initial value problem $y(0) = 2$ and all steady state solutions.
c) show concavity and show the y of all inflection points for solution curves.
d) stability analysis for all steady state solutions.

Solution: (a) g is polynomial of degree two or a quadratic. Note that the factors of g are $(y-1)$ and $(y-3)$. The graph of $g(y)$ vs. y is shown below:

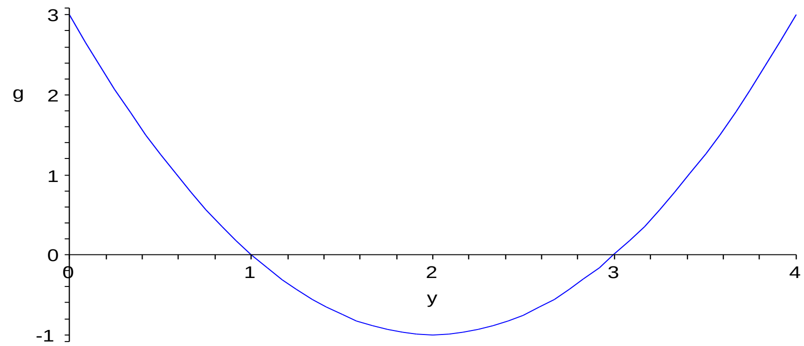

Fig. 1 $g(y)$ *vs* y

(d) Find the steady state solutions. This can be done by setting y' equal to zero in the ODE and solving for y. That is,

$$\begin{aligned} 0 &= y' \\ &= (y-1)(y-3) \end{aligned}$$

So the steady state solutions are $y = 1$ and $y = 3$. If we perturb steady state $y = 1$ a little bit to the left (a value $y < 1$), we see that y' at the point is positive. It follows that solution curve with that initial value must increase toward $y = 1$. Similarly, if we perturb steady state $y = 1$ a little bit to the right (a value $y > 1$), we see that y' at the point is negative. It follows that solution curve with that initial value must decrease toward $y = 1$. We can conclude that $y = 1$ is a stable steady state. Using this method of analysis, we see that $y = 3$ is an unstable steady state.

(b) and (c). Let us find the inflection point. This done by setting $y'' = 0$. That is,

$$\begin{aligned} 0 &= y'' \\ &= 2y - 4. \end{aligned}$$

So $y = 2$ is an inflection point. Consider the initial value problem $y(0) = 2$. Now, $y'(0) = g(y(0)) = g(2) = (2-1)(2-3) = -1 < 0$ and $y''(0) = g'(y(0)) = g'(2) = 2(2) - 4 = 0$. It follows that $y'(0)y''(0) = 0$. This yields no information about the concavity at $y = 2$; this is because $y = 2$ is an inflection point. Since at the initial value of $y(0) = 2$ ($y(t)$ is decreasing), we must chose a point a little smaller than 2 to determine the concavity. Put $y = 1.9$. It follows that $g(1.9) = (1.9-1)(1.9-3) = -0.99 < 0$ and $g'(1.9) = 2(1.9) - 4 = -0.2$. By Theorem 4.4, we can conclude that at the initial value $y(0) = 2$, $y(t)$ concave up. Because $y = 1$ is a stable steady state, the graph of y approaches $y = 1$ from above. That is,

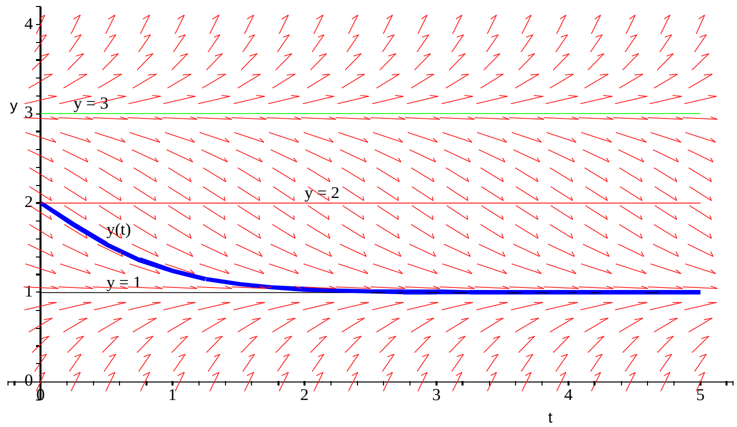

Fig. 2 $y(t)$ *vs* t

Problem 6. *Extra Credit* [5 points] Consider the ODE

$$\frac{dy}{dx} = y - 1$$

Verify that for any constant C the expression $y = Ce^x + 1$ is a solution to the ODE.

Solution:

$$\begin{aligned} y - 1 &= \frac{d}{dx}(Ce^x + 1) \\ &= Ce^x \\ &= Ce^x + 1 - 1 \\ &= y - 1 \text{ because } y = Ce^x + 1. \end{aligned}$$