EXAM II SOLUTIONS

MAT 209 · SPRING 2009

You must show all work to get full credit.

Problem 1 (25 points)**.** An ecosystem containing two species is modeled by the system of differential equations given below, where N_1 and N_2 denote the number of members of each species and the rates are annual rates of change of the species populations:

$$\begin{aligned} \frac{dN_1}{dt} &= 0.1N_1\left(1 - \frac{N_1}{100} - \frac{N_2}{50}\right) \\ \frac{dN_2}{dt} &= 0.3N_2\left(1 - \frac{N_1}{100} - \frac{N_2}{100}\right) \end{aligned}$$

(a) Determine the steady state solution of this system.
(b) Based on the above model, would you characterize the species as competitive? In the long term to which of the possible steady state solutions will the populations tend? Explain

Solution: (a) To determine the steady state solutions of this system, set $N_1' = N_2' = 0$. That is,

$$\begin{aligned} 0.1N_1\left(1 - \frac{N_1}{100} - \frac{N_2}{50}\right) &= 0 \\ 0.3N_2\left(1 - \frac{N_2}{100} - \frac{N_1}{100}\right) &= 0. \end{aligned}$$

It follows that $N_1 = 0$ and $N_1 = 100 - 2N_2$ are solutions to equation one. Now, both solutions have to also satisfy the second equation as well. Substituting zero for N_1 in equation two, we see that $N_2 = 0$ or $N_2 = 100$. So, $(0, 0)$ and $(0, 100)$ are steady states. Now, substitute $100 - 2N_2$ for N_1 in equation two. This yields $N_2 = 0$ and $N_2 = 100$. And so, $(100, 0)$ and $(0, 100)$ are also a steady states. All together, we have three steady states $(0, 0), (0, 100)$, and $(100, 0)$.

(b) We can set up a table containing the measures of inter and intraspecies competition.

TABLE 1

Species	**1**	**2**
1	0.01	0.01
2	0.02	0.01

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M. Carty, Department of Mathematics, The City College of New York.

Species 2 exerts a greater effect on species 1 than it does on itself ($0.02 > 0.01$), while species 1 exerts same pressure on itself as it does on species 2 ($0.01 \geq 0.005$). As long as both species are present to begin with, one of the species should out-compete the other species. The net effect should be the demise of species 1. The populations are likely to tend to the steady state $(0, 100)$.

Problem 2 (25 points)**.** A lake holds a population of 200,000 salmon. The natural growth rate of the salmon population is 5%. Fishing removes 30,000 salmon each year from the lake. Set up a differential equation modeling the fish population as a function of time. Solve it to find out if and when the fish population will be completely depleted.

Solution: The ODE that best models the trout population is a harvesting model. The ODE is the following:

$$\frac{dN}{dt} = 0.05N - 30000.$$

Note that the ODE is separable. Therefore,

$$\frac{1}{0.05}\ln|0.05N - 30000| = \frac{1}{0.05}\ln|u| = \frac{1}{0.05}\int \frac{\mathrm{d}u}{u} = \int \frac{\mathrm{d}N}{0.05N - 30000} = \int \mathrm{d}t = t + c.$$

We multiply both sides by 0.05, and we obtain

$$\ln|0.05N - 30000| = 0.05t + c.$$

Upon the exponentiation both sides of the equality, we get

$$|0.05N - 30000| = e^c \cdot e^{0.05t}.$$

We can deduce that general solution is

$$N(t) = 600000 + Ce^{0.06t}.$$

Note that $N(0) = 200000$. But then,

$$200000 = N(0) = 600000 - Ce^{0.05(0)} = 600000 + C.$$

So, $C = -400000$. Therefore the ODE is

$$N(t) = 600000 - 400000e^{0.05t}.$$

When the fish is completely depleted, $N = 0$. So,

$$0 = 600000 - 400000e^{0.05t}.$$

But then, $t = \frac{1}{0.05}\ln\frac{600000}{400000} = 8.11$. The salmon population will be completely depleted in approximately 8 years.

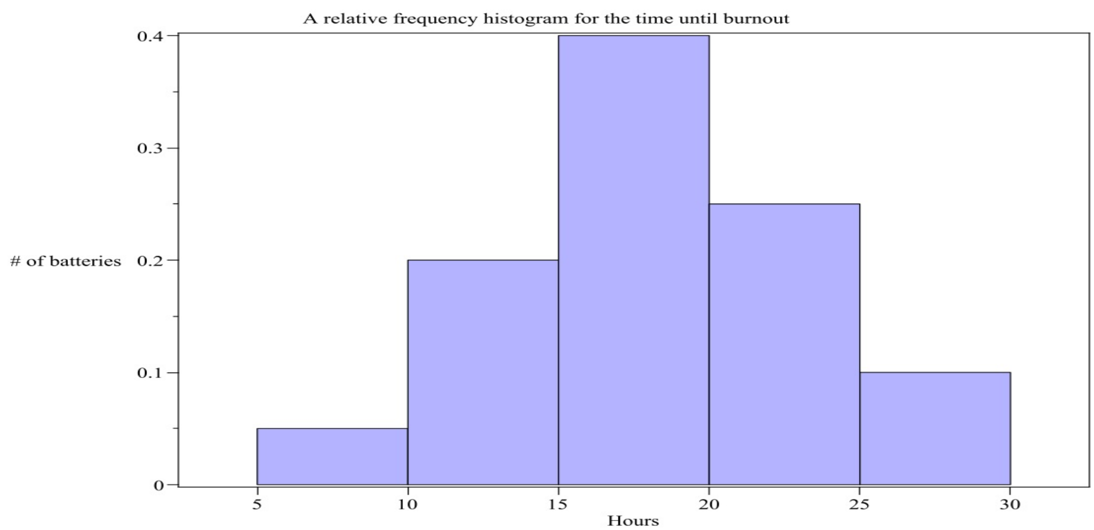
Problem 3 (25 points)**.** 100 batteries were randomly selected from a large batch and placed through a simulation of everyday use until they burned out. The table below summarizes the distribution of their lifetime (=number of hours until burn out). For example, 20 batteries lasted between a little more than 10 hours and up to 15 hours.

TABLE 2

Hours until burnout	(5, 10]	(10, 15]	(15, 20]	(20, 25]	(25, 30]
# of batteries	5	20	40	25	10

a) Based on the data, prepare a relative frequency histogram for the time until burnout for battery from the sample.
b) Based on data, estimate as accurately as you can the median number of hours that a sample battery lasted until burnout.
c) Based on the data estimate as accurately as you can the average number of hours that a sample battery lasted until burnout.

Solution: (a)



(b) We can calculate the median hours precisely by an estimation called linear interpolation. We see that 50% of the data lie in the interval $(15, 20]$. Since at the beginning of this interval we have accounted for only 25% of the values, while 65% are accounted at the end. The interpolation function is calculated as follows:

$$h(x) \quad = \quad \frac{20-15}{0.65-0.25}(x-.25)+15$$

The median occurs at $x = 0.5$. So, the median is 18.125

(c) The mean is found by adding the entries in the frequency column. We find that $n = 125$. The numerator $\overline{x}$ is obtained by adding all the data values. Ten of these values falls into the first bin $(800, 900]$. We estimated the values of these data points using the midpoint 850 of the bin interval. We proceed in similar manner for the other bin intervals. This gives the estimate

$$\overline{x} = \frac{(5)(7.5) + (20)(12.5) + (40)(17.5) + (25)(22.5) + (25)(27.5)}{100} = \frac{1825}{100} = 18.25.$$

Problem 4 (20 points)**.**

(a) Consider the four points $(8, 17), (2, 5), (6, 13)$, and $(4, 9)$. Find the quantities s_x, s_y, c_{xy}, and r and use these to find the regression line.
(b) [5 points] What estimate would you give for the value of y when $x = 3$?

Solution: Recall that

$$\begin{aligned} \hat{b} &= \frac{c_{xy}}{s_x^2} \text{ is the slope of the regression line.} \\ \hat{a} &= \overline{y} - \hat{b}\overline{x} \text{ is the } y\text{-intercept of the regression line.} \end{aligned}$$

Now, we must determine $\overline{x}, \overline{y}, c_{xy}, r, s_y$, and s_x. So,

$$\begin{aligned} \overline{x} &= \frac{8+2+6+4}{4} = 5 \\ \overline{y} &= \frac{17+5+13+9}{4} = 11 \\ c_{xy} &= \frac{(8-5)(17-11) + (2-5)(5-11) + (6-5)(13-11) + (4-5)(9-11)}{3} = 13.33 \\ s_x &= \sqrt{\frac{(8-5)^2 + (2-5)^2 + (6-5)^2 + (4-5)^2}{3}} = 2.58 \\ s_y &= \sqrt{\frac{(17-11)^2 + (5-11)^2 + (13-11)^2 + (9-11)^2}{3}} = 5.16 \\ r &= \frac{13.33}{(2.58)(5.16)} = 1. \end{aligned}$$

So $\hat{b} = 2$. Now,

$$\hat{a} = 11 - (2)(5) = 1.$$

Therefore the equation for the regression line is $y = 2x + 1$. (b) The predicted value of y at $x = 3$ is 7.

Problem 5. (Extra Credit)[10 points] You purchase a new car and keep it for 10 years. Every year you record the amount you spend on repairs, including routine maintenance. Which of the following would you expect for the correlation coefficient?

(a) moderately positive
(b) close to zero
(c) moderately negative
(d) zero

Solution:*Ans.* (a) Moderately positive. As the car gets older it will require more maintenance. However, the expenditures may rise more rapidly with the car's age so a non-linear pattern would probably better describe the observed relationship.