PRACTICE EXAM I

MAT 209 · SPRING 2009

You must show all work to get full credit.

Problem 1 (10 points)**.** Solve the differential equation

$$\frac{dy}{dx} = xy$$

subject to the initial condition $y(0) = 2$.

Solution: The differential equation is separable since the right side of the expression is a product the dependent and independent variable. Thus,

$$\ln|y| = \int \frac{dy}{y} = \int x\mathrm{d}x = \frac{1}{2}x^2 + C.$$

Exponentiating both sides of the equality, we get

$$|y| = e^{\ln|y|} = e^{\frac{1}{2}x^2 + C} = e^C e^{\frac{1}{2}x^2}.$$

Because $|y| = \pm y$, it follows that

$$\begin{aligned} y(x) &= \pm e^C e^{\frac{1}{2}x^2} \\ &= Ce^{\frac{1}{2}x^2}. \end{aligned}$$

Since we have found the general solution, we can now solve for the initial value problem.

$$\begin{aligned} 2 &= y(0) \\ &= Ce^{\frac{1}{2}(0)^2} \\ &= Ce^0 \\ &= C. \end{aligned}$$

But then,

$$y(x) = 2e^{\frac{1}{2}x^2}.$$

Problem 2 (10 points)**.** Solve the differential equation

$$\frac{dy}{dx} = \frac{2y}{x+1}$$

subject to the initial condition $y(0) = 3$.

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Solution: The differential equation is separable since the right side of the expression is a quotient with the dependent in numerator and the independent variable in denominator. Thus,

$$\ln|y| = \int \frac{dy}{y} = \int \frac{2}{x+1}\mathrm{d}x = 2\ln|x+1| + C = \ln(x+1)^2 + C.$$

Exponentiating both sides of the equality, we get

$$|y| = e^{\ln|y|} = e^{\ln(x+1)^2 + C} = e^C(x+1)^2.$$

Because $|y| = \pm y$, it follows that

$$\begin{aligned} y(x) &= \pm e^C(x+1)^2 \\ &= C(x+1)^2. \end{aligned}$$

Since we have found the general solution, we can now solve for the initial value problem.

$$\begin{aligned} 3 &= y(0) \\ &= C(0+1)^2 \\ &= C. \end{aligned}$$

But then,

$$y(x) = 3(x+1)^2.$$

Problem 3 (10 points)**.** Consider the initial value problem

$$\frac{dy}{dx} = x^3 - y$$

where $y(0) = 1$. Use the Eulers method with step size 0.25 to find the approximate value of $y(1)$. (Compute your answer to 3 decimal places.)

Solution: Note that $\Delta x = (b-a)/n$. Since the step size is 0.25, $a = 0$, and $b = 1$; it must be that $n = 4$. We must have four subintervals. Recall Euler's formula:

$$y(x + \Delta x) \approx y(x) + y'(x)\Delta x.$$

We will rewrite this expression as

$$y_{n+1} = y_n + y_n'\Delta x.$$

For $n = 0$, it follows that

$$y_1 = y_0 + y_0'\Delta x,$$

where $y_0 = y(0) = 1, y_0' = x_0^3 - y_0, \Delta x = 0.25, x_0 = 0$. So

$$\begin{aligned} y_1 &= 1 + (0^3 - 1)(0.25) \\ &= 0.75. \end{aligned}$$

For $n = 1$, we have

$$\begin{aligned} y_2 &= y_1 + y_1'\Delta x \\ &= y_0 + (x_1^3 - y_1)\Delta x \\ &= (0.75) + ((0.25)^3 - (0.75))(0.25) \\ &= 0.5664. \end{aligned}$$

For $n = 2$, we get

$$\begin{aligned} y_3 &= y_2 + y_2' \Delta x \\ &= y_2 + (x_2^3 - y_2)\Delta x \\ &= (0.5664) + ((0.50)^3 - (0.5664))(0.25) \\ &= 0.4561. \end{aligned}$$

For $n = 3$, we have

$$\begin{aligned} y_4 &= y_3 + y_3' \Delta x \\ &= y_2 + (x_2^3 - y_2)\Delta x \\ &= (0.4561) + ((0.75)^3 - (0.4561))(0.25) \\ &= 0.448. \end{aligned}$$

Recall that $y_4 = y(1)$. Therefore, $y(1) = 0.448$.

Problem 4 (10 points)**.** Use geometric analysis to analyze the differential equation

$$\frac{dy}{dx} = -y^2 + 9y - 14$$

Your answer should include:

a) a graph of $g(y)$ vs. y , where $g(y)$ is the derivative of y as a function of y.
b) sketches of the solution curves $y(t)$ for the initial value problem $y(0) = 3$ and all steady state solutions.
c) show concavity and show the y of all inflection points for solution curves.
d) stability analysis for all steady state solutions.

Solution: (a) g is polynomial of degree two or a quadratic. Note that the factors of g are $(2 - y)$ and $(y - 7)$. The graph of $g(y)$ vs. y is shown below:

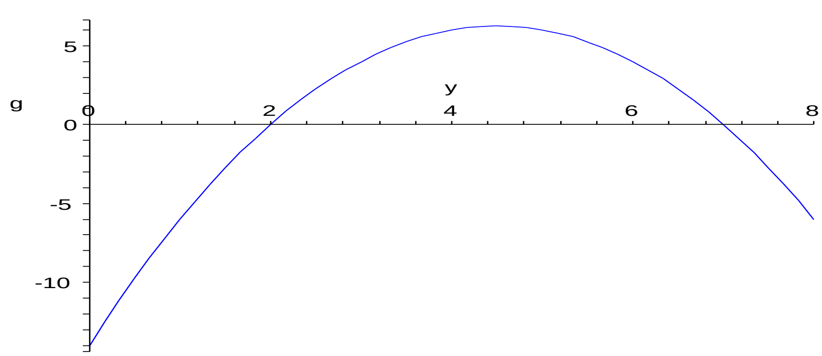

Fig. 1 $g(y)$ *vs* y

(d) Find the steady state solutions. This can be done by setting y' equal to zero in the ODE and solving for y. That is,

$$\begin{aligned} 0 &= y' \\ &= -y^2 + 9y - 14 \\ &= -(y^2 - 9y + 14) \\ &= -(y-2)(y-7) \end{aligned}$$

So the steady state solutions are $y = 2$ and $y = 7$. If we perturb steady state $y = 2$ a little bit to the left (a value of $y < 2$), we see that y' at the point is negative. It follows that solution curve with that initial value must decrease away from $y = 2$. Similarly, if we perturb steady state $y = 2$ a little bit to the right (a value of $y > 2$), we see that y' at the point is positive. It follows that solution curve with that initial value must increase away from $y = 2$. We can conclude that $y = 2$ is an unstable steady state. Using this method of analysis, we see that $y = 7$ is a stable steady state.

(b) and (c). Let us find the inflection point. This done by setting $y'' = 0$. That is,

$$\begin{aligned} 0 &= y'' \\ &= -2y + 9. \end{aligned}$$

So $y = 4.5$ is an inflection point. Consider the initial value problem $y(0) = 3$. Now, $y'(0) = g(y(0)) = g(3) = -(3-2)(3-7) = 4 > 0$ and $y''(0) = g'(y(0)) = g'(3) = -2(3) + 9 = 3 > 0$. It follows that $y'(0)y''(0) = 4(3) = 7 > 0$. By Theorem 4.4, we can conclude that at the initial value $y(0) = 3$, y concave up. When the graph of y passing beyond $y = 4.5$ (the inflection point), y concave down. Because $y = 7$ is a stable steady state, the graph of y approaches $y = 7$ from below. That is,

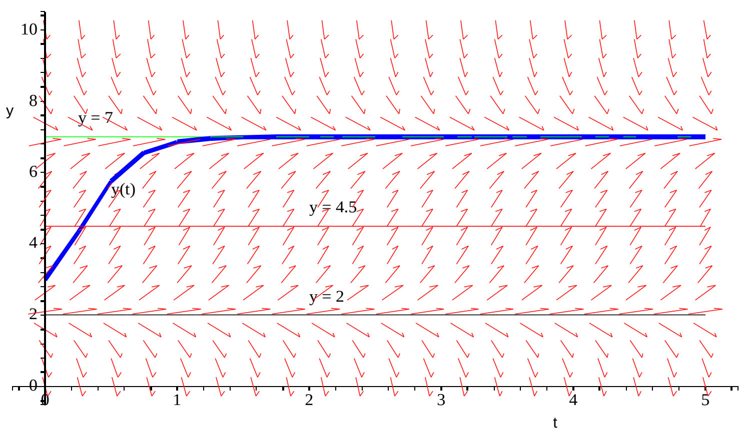

Fig. 2 $y(t)$ *vs* t