SUPPLEMENTAL PROBLEMS

MAT 209 · SPRING 2009

You must show all work to get full credit.

Problem 1 (10 points)**.** A lake holds a population of 100,000 trout. The natural growth rate of the trout population is 6%. Fishing removes 20,000 trout each year from the lake. Set up a differential equation modeling the fish population as a function of time. Solve it to find out if and when the fish population will be completely depleted.

Solution: The ODE that best models the trout population is a harvesting model. The ODE is the following:

$$\frac{dN}{dt} = 0.06N - 20000.$$

Note that the ODE is separable. Therefore,

$$\frac{1}{0.06}\ln|0.06N - 20000| = \frac{1}{0.06}\ln|u| = \frac{1}{0.06}\int\frac{\mathrm{d}u}{u} = \int\frac{\mathrm{d}N}{0.06N - 20000} = \int \mathrm{d}t = t + c.$$

We multiply both sides by 0.06, and we obtain

$$\ln|0.06N - 20000| = 0.06t + c.$$

Upon the exponentiation both sides of the equality, we get

$$|0.06N - 20000| = e^c \cdot e^{0.06t}.$$

We can deduce that general solution is

$$N(t) = 333333 - Ce^{0.06t}.$$

Note that $N(0) = 100000$. But then,

$$100000 = N(0) = 333333 - Ce^{0.06(0)} = 333333 - C.$$

So, $C = 233333$. Therefore the ODE is

$$N(t) = 333333 - 233333e^{0.06t}.$$

When the fish is completely depleted, $N = 0$. So,

$$0 = 333333 - 233333e^{0.06t}.$$

But then, $t = \frac{1}{0.06}\ln\frac{333333}{233333} = 5.94$. The trout population will be completely depleted in approximately 6 years.

Date: April 29, 2009 *Due Date*: Monday, April 27, 2009.

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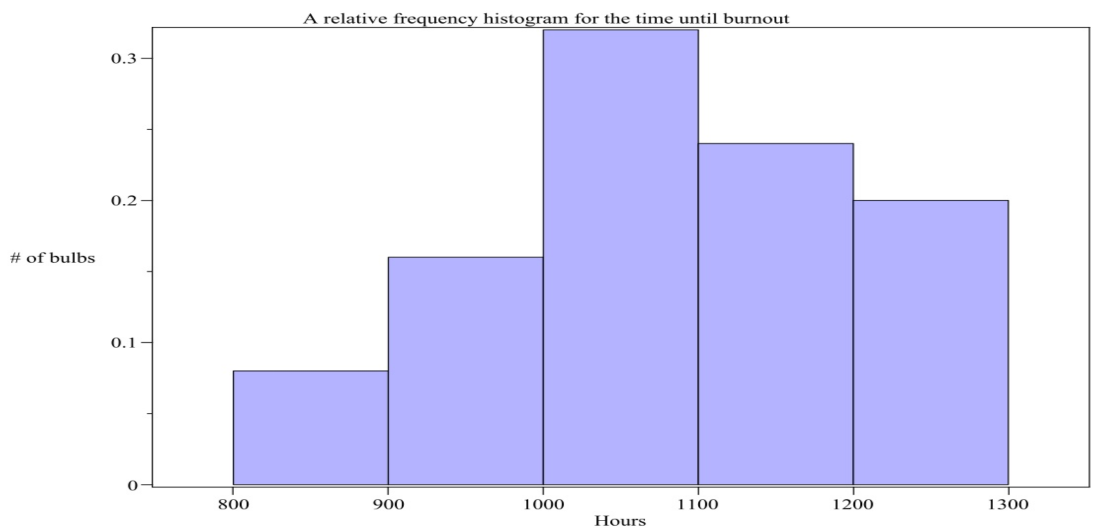
Problem 2 (10 points)**.** 125 light bulbs were randomly selected from a large batch and placed through a simulation of everyday use until they burned out. The table below summarizes the distribution of their lifetime (=number of hours until burn out). For example, 20 bulbs lasted between a little more than 900 hours and up to 1000 hours.

TABLE 1

Hours until burnout	(800, 900]	(900, 1000]	(1000,1100]	(1100,1200]	(1200,1300]
# of bulbs	10	20	40	30	25

a) Based on the data, prepare a relative frequency histogram for the time until burnout for bulbs from the sample.
b) Based on data, estimate as accurately as you can the median number of hours that a sample bulb lasted until burnout.
c) Based on the data estimate as accurately as you can the average number of hours that a sample bulb lasted until burnout.

Solution: (a)



(b) We can calculate the median hours precisely by an estimation called linear interpolation. We see that 55% of the data lie in the interval $(1000, 1100]$. Since at the beginning of this interval we have accounted for only 24% of the values, while 55% are accounted at the end. The interpolation function is calculated as follows:

$$\begin{aligned} h(x) &= \frac{1100 - 1000}{0.56 - 0.24}(x - .24) + 1000 \\ &= 312.5x + 925. \end{aligned}$$

The median occurs at $x = 0.5$. So, the median is 1081.25

(c) The mean is found by adding the entries in the frequency column. We find that $n = 125$. The numerator $\overline{x}$ is obtained by adding all the data values. Ten of these values falls into the first bin $(800, 900]$. We estimated the values of these data points using the midpoint 850 of the bin interval. We proceed in similar manner for the other bin intervals. This gives the estimate

$$\overline{x} = \frac{(10)(850) + (20)(950) + (40)(1050) + (30)(1150) + (25)(1250)}{125} = \frac{135250}{125} = 1082.$$

Problem 3 (10 points)**.** An ecosystem containing two species is modeled by the system of differential equations given below, where N_1 and N_2 denote the number of members of each species and the rates are annual rates of change of the species populations:

$$\begin{aligned} \frac{dN_1}{dt} &= 0.1N_1\left(1 - \frac{N_1}{60} - \frac{N_2}{60}\right) \\ \frac{dN_2}{dt} &= 0.2N_2\left(1 - \frac{N_2}{90} - \frac{N_1}{30}\right) \end{aligned}$$

a) Find all steady-state solutions of this system.
b) Based on the above model, would you characterize the species as competitive? In the longterm to which of the possible steady state solutions will the populations tend? Explain

Solution: (a) To determine the steady state solutions of this system, set $N_1' = N_2' = 0$. That is,

$$\begin{aligned} 0.1N_1\left(1 - \frac{N_1}{60} - \frac{N_2}{60}\right) &= 0 \\ 0.2N_2\left(1 - \frac{N_2}{90} - \frac{N_1}{30}\right) &= 0. \end{aligned}$$

It follows that $N_1 = 0$ and $N_2 = 60 - N_1$ are solutions to equation one. Now, both solutions have to also satisfy the second equation as well. Substituting zero for N_1 in equation two, we see that $N_2 = 0$ or $N_2 = 90$. So, $(0, 0)$ and $(0, 90)$ are steady states. Now, substitute $60 - N_1$ for N_2 in equation two. This yields $N_1 = 60$ and $N_1 = 15$. And so, $(60, 0)$ and $(15, 45)$ are also a steady states.

(b) We can set up a table containing the measures of inter and intraspecies competition.

Table 2

Species	**1**	**2**
1	0.0167	0.033
2	0.0167	0.011

Species 1 exerts a greater effect on species 2 than it does on itself ($0.033 > 0.0167$), while species 2 exerts a greater pressure on species 1 than it does on itself ($0.0167 > 0.011$). As long as both species are present to begin with, one of the species should out-compete the other species. The net effect should be the demise of one of the species. The populations are likely to tend to the steady states $(60, 0)$ and $(0, 90)$.