

# A CONSTRUCTION OF ARITHMETIC PROGRESSION-FREE SEQUENCES AND ITS ANALYSIS

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ABSTRACT. We describe a particular greedy construction of an arithmetic progression-free sequence from a finite composition. We also give an analysis on the properties of the resulting sequence.

## 1. INTRODUCTION

One of the many open problems that remain in number theory addresses the question of existence of arbitrarily long finite arithmetic progressions of primes [3]. Erdős asked a more general question: If  $A$  is an infinite set of positive integers such that the series  $\sum_{a \in A} a^{-1}$  diverges, then must  $A$  contain arbitrarily long finite arithmetic progressions [4]? If the answer to his question is yes, then this would definitely imply the existence of arbitrarily long finite arithmetic progressions of primes since  $\sum_p p^{-1}$  is known to diverge.

The main concern of this paper is approaching Erdős' conjecture by the contrapositive: If  $A$  does not contain arbitrarily long finite arithmetic progressions, does  $\sum_{a \in A} a^{-1}$  necessarily converge? More specifically, we mainly concentrate on a specific case of his conjecture; i.e., we concentrate on sets that do not contain *any* A.P. Hence it suffices to concentrate on sequences without a 3-term A.P. (A.P. will denote arithmetic progression from this point forward). We should also make the note that *progression*, without the qualifier arithmetic, will refer to a finite sequence. In the case that an A.P. is infinite, we will say A.P. sequence. We develop a construction to produce A.P.-free sequences; we then analyze and offer examples of the construction.

## 2. DESCRIPTION OF THE CONSTRUCTION

**2.1. Introduction.** In order to discuss our construction, we must first define the notion of an irreducible composition.

**Definition 1.** A composition,  $m = c_1 + c_2 + \dots + c_k$ , is reducible when the following condition is satisfied:

$$c_i = c_{i + \frac{k+1}{2}} \quad \text{for } i = 1, \dots, \frac{k-1}{2} \quad \text{and for } k \text{ odd.}$$

**Example 2.** The composition 1, 2, 1 is reducible while the composition 1, 2, 4, 2, 1 is irreducible.

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The basic idea is to start with an irreducible composition,  $c_1, \dots, c_{n-1}$  (we will list a composition as before instead of using the convention  $c_1 + \dots + c_{n-1}$ ), that satisfies the following conditions.

**Proposition 3.** *A progression  $S$  constructed from a composition,  $k = c_1 + c_2 + \dots + c_n$ , such that  $d_1 = 1, d_{n+1} = \sum_{1 \leq i \leq n} c_i$ , is an A.P.-free progression if and only if  $\sum_{a \leq i \leq s} c_i \neq \sum_{s < j \leq b} c_j$ , for all  $a, b, s, 1 \leq a \leq s < b \leq n$ .*

*Proof.* The proof is straightforward and therefore omitted.  $\square$

Upon starting with an irreducible composition, an exact copy of the composition is appended leaving an open spot for a new value,  $\mu$ , as in the following illustration:

$$\begin{array}{c} c_1, \dots, c_{n-1} \\ c_1, \dots, c_{n-1}, \mu, \text{copy} \\ c_1, \dots, c_{n-1}, \mu, c_1, \dots, c_{n-1}. \end{array}$$

The value of  $\mu$  is then chosen so that the resulting composition still satisfies the conditions of Proposition 3. This process is then repeated indefinitely.

Notice that after one iteration we have doubled the number of elements in the sequence and then added one more. We can also re-index the resulting sequence as follows:

$$\begin{array}{c} d_1, \dots, d_{n-1}, \mu, d_1, \dots, d_{n-1} \\ d_1, \dots, d_{n-1}, \mu = d_n, d_{n+1}, \dots, d_{2n-1}. \end{array}$$

**2.2. Formal Description of the Construction.** We now formally describe the process.

Let  $c_1, \dots, c_{n-1}$  be positive integers,  $n \geq 2$  such that

$$\sum_{a \leq i \leq s} c_i \neq \sum_{s < j \leq b} c_j, \quad \text{for all } 1 \leq a \leq s < b < n$$

The conditions above are equivalent to those in Proposition 3. Hence by Proposition 3, the finite set of integers  $S = \{m_k | k < n, m_1 = 1 \text{ and } m_k = \sum_{1 < i \leq k} c_i\}$  also contains no A.P. We now inductively define a sequence  $\{d_k\}$  of positive integers by

$$d_k = \begin{cases} c_k, & \text{if } 1 \leq k < n \\ \mu_t, & \text{if } k = 2^t n \text{ for some } t \geq 0 \\ d_{k-2^t n}, & \text{if } 2^t < k < 2^{t+1} n, \end{cases}$$

where  $\mu_t$  is the least positive integer such that for all  $1 \leq a, b < 2^{t+1} n$ ,

$$\begin{aligned} (1) \quad \mu_t &\neq \sum_{s \leq j \leq b} d_j - \sum_{2^t n + 1 \leq i < s} d_i - \sum_{a \leq i < 2^t n} d_i \quad \text{for all } 2^t n + 1 \leq s \leq b \\ (2) \quad \mu_t &\neq \sum_{a \leq i \leq s} d_j - \sum_{2^t n + 1 \leq j \leq b} d_j - \sum_{s < j < 2^t n} d_j \quad \text{for all } a \leq s < 2^t n. \end{aligned}$$

Notice that in 1 and 2, the RHS takes only a finite set of possible values, and so by the Well-Ordering principle,  $\mu_t$  is well-defined. We should also remark here that this algorithm produces sequences that coincide with the greedy construction of Odlyzko in certain cases.

It is now necessary to show that any sequence constructed as above does not contain an A.P.

**Theorem 4.** Let  $\{d_k\}$  be a sequence constructed as above from  $c_1, \dots, c_{n-1}$ . Then

(1) For any  $1 \leq a \leq s < b$ ,

$$\sum_{a \leq i \leq s} d_i \neq \sum_{s < j \leq b} d_j.$$

(2) The subsequence  $d_{2^0 n}, d_{2^1 n}, d_{2^2 n}, \dots$  is strictly increasing.

*Proof.* We will prove 1 by induction. It is true for  $1 \leq a \leq s < b < 2^0 n$  by assumption, so suppose the result is true for all  $1 \leq a \leq s < b < 2^t n$ . By construction of  $\{d_k\}$  and the induction hypothesis, the result holds for  $2^t n < a \leq s < b < 2^{t+1} n$ , so it remains only to show that it holds for  $1 \leq a \leq s, 2^t n < b < 2^{t+1} n$ . First suppose that  $s < 2^t n$ . Then

$$\begin{aligned} \sum_{a \leq i \leq s} d_i - \sum_{s < j \leq b} d_j &= \sum_{a \leq i \leq s} d_i - \sum_{s < j < 2^t n} d_j - \sum_{2^t n < j \leq b} d_j - \mu_t \\ &= \sum_{a \leq i \leq s} d_i - \sum_{2^t n + 1 \leq j \leq b} d_j - \sum_{s < j < 2^t n} d_j - \mu_t \\ &\neq 0 \end{aligned}$$

since  $\mu_t$  satisfies 2.

Now suppose that  $s \leq 2^t n$ . Then,

$$\begin{aligned} \sum_{a \leq i \leq s} d_i - \sum_{s < j \leq b} d_j &= \sum_{a \leq i < 2^t n} d_i + \mu_t + \sum_{2^t n < i \leq s} d_i - \sum_{s < j \leq b} d_j \\ &= \sum_{a \leq i < 2^t n} d_i + \mu_t + \sum_{2^t n + 1 \leq i \leq s} d_i - \sum_{s < j \leq b} d_j \\ &\neq 0 \end{aligned}$$

since  $\mu_t$  satisfies 1.

We now prove 2. Notice that  $\mu_{t+1} \geq \mu_t$  by definition of  $\mu_t$  since  $\mu_{t+1}$  must in particular satisfy the same inequalities as does  $\mu_t$ . To see this, notice that  $\mu_{t+1}$  cannot be less than  $\mu_t$  since  $\mu_t$  is the least positive integer such that 1 and 2 hold. I.e.,  $\mu_{t+1} < \mu_t$  contradicts the definition of  $\mu_t$ . Thus it suffices to show that  $\mu_{t+1} \neq \mu_t$ . This follows immediately from part 1 by taking  $a = 1, s = 2^t n, b = 2^{t+1} n$ :

$$\begin{aligned} 0 &\neq \sum_{1 \leq i \leq 2^t} d_i - \sum_{2^t n < j \leq 2^{t+1} n} d_j \\ &= \sum_{1 \leq i < 2^t} d_i + \mu_t - \sum_{2^t n < j < 2^{t+1} n} d_j - \mu_{t+1} \\ &= \sum_{1 \leq i < 2^t} d_i + \mu_t - \sum_{1 \leq j < 2^t n} d_j - \mu_{t+1} \\ &= \mu_t - \mu_{t+1}. \end{aligned}$$

□

**Corollary 5.** Let  $c_1, \dots, c_{n-1}$  and  $\{d_k\}$  be defined as above. Then the sequence defined by  $m_1 = 1, m_{k+1} = m_k + d_k$  contains no arithmetic progressions.

*Proof.* Proof follows immediately from Proposition 3 since the conditions in Theorem 4 are equivalent to those in Proposition 3. □

## 3. EXAMPLE OF THE CONSTRUCTION

Let us start with the initial composition of 1. Although it may seem rather uninteresting at first glance, some very neat observations and patterns arise. In fact, starting the construction with the initial composition 1 coincides with the greedy construction of  $S(1)$  described by Odlyzko and Stanley [1].

By expanding the partition 1 by the algorithm we get the following partition after four iterations:

1, 2, 1, 5, 1, 2, 1, 14, 1, 2, 1, 5, 1, 2, 1, 41, 1, 2, 1, 5, 1, 2, 1, 14, 1, 2, 1, 5, 1, 2, 1

The value of the  $\mu$ 's in order are 2, 5, 14, 41. We then set up the equations as follows:

$$\begin{aligned}\mu_1 - \mu_0 &= 5 - 2 = 3^1 \\ \mu_2 - \mu_1 &= 14 - 5 = 3^2 \\ \mu_3 - \mu_2 &= 41 - 14 = 3^3 \\ &\vdots \\ \mu_{t+1} - \mu_t &= 3^{t+1}.\end{aligned}$$

Then by adding all the equations, we get

$$\begin{aligned}\mu_{t+1} - \mu_0 &= 3(1 + \cdots + 3^t) \\ &= 3\left(\frac{3^{t+1} - 1}{2}\right) \\ &= \frac{3^{t+2} - 3}{2} \\ \mu_{t+1} &= \frac{3^{t+2} - 3}{2} + 2 \\ &= \frac{3^{t+2} + 1}{2}.\end{aligned}$$

Now, consider the following (remember that we are using the initial partition 1, hence  $n = 2$ ):

$$\begin{aligned}m_1 &= 1 \\ m_2 &= 1 + d_1 \\ &\vdots \\ m_{n-1} &= 1 + d_1 + \cdots + d_{n-2} \\ m_n &= 1 + d_1 + \cdots + d_{n-2} + d_{n-1} = \mu_0 \\ m_{n+1} &= 1 + d_1 + \cdots + d_n \\ &\vdots \\ m_{2n} &= 1 + d_1 + \cdots + d_{n-1} + d_n + d_{n+1} + \cdots + d_{2n-1} = \mu_1 \\ &\vdots\end{aligned}$$

Hence  $m_k$  is the A.P.-free sequence. Now since each  $\mu_t$  can now be described by a function of  $t$ , each  $m_{a=2^t n}$  can also be described. Therefore, we now have the means to determine the convergence properties of the sum of the reciprocals.

We now show the convergence of the sum of the reciprocals of the sequence constructed by the greedy algorithm. Then we have

$$\begin{array}{ccc}
m_1 \geq 1 & \text{implies} & 1 \leq 1 \\
\underbrace{m_2, m_3}_{2^1 \text{ terms}} \geq \mu_0 & \text{implies} & \sum_{i=2}^{2^{n-1}} m_i^{-1} \leq \frac{2^0 \cdot 2}{3^{0+1} + 1} \\
& & \vdots \\
\underbrace{m_{2^{2n}}, \dots, m_{2^{2+1}n-1}}_{2^{2n} \text{ terms}} \geq \mu_t & \text{implies} & \sum_{i=2^{2n}}^{2^{2+1}n-1} m_i^{-1} \leq \frac{2^t \cdot 2}{3^{t+1} + 1}
\end{array}$$

Thus adding the inequalities in the last column and replacing  $n$  with 2 we get

$$1 + \sum_{t=0}^{\infty} \frac{2^{t+1} \cdot 2}{3^{t+1} + 1} = 1 + 4 \sum_{t=0}^{\infty} \frac{2^t}{3^{t+1} + 1} \geq \sum_{t=1}^{\infty} \frac{1}{m_t}$$

The summation  $\sum_{t=1}^{\infty} \frac{2^t}{3^{t+1} + c}$  clearly converges.

#### 4. CONSEQUENCES OF THE CONSTRUCTION

In the last section, it was shown that the  $\mu_t$ 's for the greedy algorithm followed a certain pattern. This is not, however, the general pattern for all sequences created by our construction. We now attempt to generalize a pattern for all  $\mu_t$ 's with any initial composition.

**Lemma 6.** *Let  $\{d_m\}$  be a sequence constructed as above from  $c_1, \dots, c_{n-1}$ . Then for any  $T > t \geq 0$ , we have:*

- (1)  $d_i = d_{i+2^T n + 2^{2t} n}$ , for all  $1 \leq i < 2^{2t} n$
- (2)  $d_i = d_{i+2^{t+1} n}$ , for all  $2^T n - 2^{2t} n \leq i < 2^T n$
- (3)  $d_j = d_{j-2^{t+1} n}$ , for all  $2^T n < j < 2^T n + 2^{2t} n$
- (4)  $d_j = d_{j+2^{2t} n}$ , for all  $2^T n - 2^{t+1} n - 2^{2t} n \leq j \leq 2^T n - 2^{t+1} n$

*Proof.* Notice that 1 follows immediately from the definition of  $\{d_n\}$  since  $d_{i+2^T n + 2^{2t} n} = d_{i+2^{2t} n} = d_i$ .

For 2, notice that we may write  $i = 2^T n - 2^{2t} n + j$  for  $0 \leq j < 2^{2t} n$ . Then  $i + 2^{t+1} n = 2^T n + 2^{2t} n + j$  for  $0 \leq j < 2^{2t} n$ . So,  $d_{i+2^{t+1} n} = d_{2^T n + 2^{2t} n + j} = d_j$  by 1. And then  $d_i = d_{2^{2T-1} n + \dots + 2^{2t} n + j} = d_{2^{2T-2} n + \dots + 2^{2t} n + j} = \dots = d_j$ .

To prove 3, let  $i = j - 2^{t+1} n$ . Then  $2^T n < j < 2^T n + 2^{2t} n$  implies that  $2^T n - 2^{t+1} n < i < 2^T n$ .

Finally, for 4, notice that from 2 we can write  $d_i = d_{j+2^{t+1} n} = d_{j-2^{t+1} n} = d_{j-2^{2t} n}$  for  $2^T n - 2^{2t} n \leq j < 2^T n$ . Then for  $i = 2^{t+1} n$ , we have  $d_i = d_{2^{2t} n + i}$  for  $2^T n - 2^{t+1} n - 2^{2t} n \leq j \leq 2^T n - 2^{t+1} n$ .  $\square$

**Theorem 7.** *Let  $T > t + 1$ . If  $\mu_T > \sum_{1 \leq i < 2^{2t} n} d_i$ , then  $\mu_T > \sum_{1 \leq i \leq 2^{2t} n} d_i$ .*

*Proof.* Assume by contradiction that  $\mu_T < \sum_{1 \leq i < 2^{2t} n} d_i$ . Let  $\Delta = \sum_{1 \leq i \leq 2^{2t} n} d_i - \mu_T > 0$ .

We now will show that  $\mu_t - \Delta = \mu_T - \sum_{1 \leq i < 2^{2t} n} d_i$  satisfies 1 and 2 thus contradicting the minimality of  $\mu_t$ . We now proceed by cases.

Case 1: Now suppose, by way of contradiction, that there exist  $1 \leq a < 2^t n < b < 2^{t+1} n$  and  $s \in [a, 2^t n]$  so that  $\sum_{a \leq i \leq s} d_i = \sum_{s < j < 2^t n} d_j + \mu_t - \Delta + \sum_{2^t n < j \leq b} d_j$ . We then

have

$$(3) \quad \sum_{a \leq i \leq s} d_i = \sum_{s < j < 2^t n} d_j + \mu_T - \sum_{1 \leq i < 2^t n} d_i + \sum_{2^t n < j \leq b} d_j.$$

Now let  $\tilde{a} = 2^T n - 2^t n + a$ ,  $\tilde{b} = 2^T n - 2^t n + b$  and  $\tilde{s} = 2^T n - 2^t n + s$ . Then substituting into equation 3, we have,

$$\sum_{\tilde{a} \leq i \leq \tilde{s}} d_i = \sum_{\tilde{s} < j < 2^T n} d_j + \mu_T - \sum_{1 \leq i < 2^t n} d_i + \sum_{2^t n < j \leq \tilde{b}} d_j.$$

Which upon simplifying, we get

$$(4) \quad \sum_{\tilde{a} \leq i \leq \tilde{s}} d_i + \sum_{1 \leq i < 2^t n} d_i = \sum_{\tilde{s} < j \leq \tilde{b}} d_j.$$

We now have  $2^T n - 2^t n + 1 \leq \tilde{a} < \tilde{b} < 2^T n + 2^t n$ ,  $\tilde{s} \in [\tilde{a}, 2^T n]$ . From Lemma 6, we have the following identities:

$$(5) \quad \sum_{1 \leq i < 2^t n} d_i = \sum_{2^T n - 2^{t+1} n + 1 \leq i < 2^T n - 2^t n} d_i$$

$$(6) \quad \sum_{2^T n - 2^t n \leq i < \tilde{a}} d_i = \sum_{2^T n + 2^t n \leq i < \tilde{a} + 2^{t+1} n} d_i$$

$$(7) \quad \sum_{\tilde{b} - 2^{t+1} n < i < 2^T n - 2^t n} d_i = \sum_{\tilde{b} < j < 2^T n + 2^t n} d_j$$

$$(8) \quad \sum_{\tilde{b} - 2^{t+1} n < i < 2^T n - 2^t n} d_i = \sum_{\tilde{b} - 2^{t+1} n - 2^t n < i < 2^T n - 2^{t+1} n} d_i$$

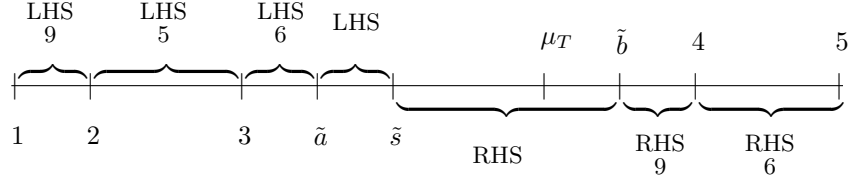
Notice that from Equation 7 and Equation 8, we have

$$(9) \quad \sum_{\tilde{b} - 2^{t+1} n - 2^t n < i < 2^T n - 2^{t+1} n} d_i = \sum_{\tilde{b} < j < 2^T n + 2^t n} d_j.$$

By substituting Equation 5 into equation 4 and adding the LHS of Equation 6 and Equation 9 to the LHS of equation 4 while adding the RHS of Equation 6 and Equation 9 to the RHS of equation 4, and we get the following

$$(10) \quad \begin{aligned} & \sum_{\tilde{a} \leq i \leq \tilde{s}} d_i + \sum_{2^T n - 2^{t+1} n < i < 2^T n - 2^t n} d_i + \sum_{2^T n - 2^t n \leq i < \tilde{a}} d_i + \sum_{\tilde{b} - 2^{t+1} n - 2^t n < i < 2^T n - 2^{t+1} n} d_i \\ & = \sum_{\tilde{s} < j \leq \tilde{b}} d_j + \sum_{2^T n + 2^t n \leq j < \tilde{a} + 2^{t+1} n} d_j + \sum_{\tilde{b} < j < 2^T n + 2^t n} d_j. \end{aligned}$$

We use the following picture to illustrate the substitutions:



where

- (1)  $\tilde{b} - 2^{t+1}n$
- (2)  $2^T n - 2^{t+1}n + 1$
- (3)  $2^T n - 2^t n$
- (4)  $2^T n + 2^t n$
- (5)  $2^T n + 2^t n + a$ .

Then by simplifying Equation 10, we have

$$\sum_{\tilde{b}-2^{t+1}n-2^t n < i \leq \tilde{s}} d_i = \sum_{\tilde{s} < j < \tilde{a}+2^{t+1}n} d_j,$$

contradicting the definition of  $\mu_T$ .

Case 2: Now suppose, by way of contradiction, that there exist  $1 \leq a < 2^t n < b < 2^{t+1}n$  and  $s \in [2^t n, b]$  so that  $\sum_{a \leq i < 2^t n} d_i + \mu_t - \Delta + \sum_{2^t n < i \leq s} d_i = \sum_{s < j < b} d_j$ . We then

have

$$(11) \quad \sum_{a \leq i < 2^t n} d_i + \mu_T - \sum_{1 \leq j < 2^t n} d_j + \sum_{2^t n < i \leq s} d_i = \sum_{s < j \leq b} d_j$$

Now let  $\tilde{a} = 2^T n - 2^t n + a$ ,  $\tilde{b} = 2^T n - 2^t n + b$  and  $\tilde{s} = 2^T n - 2^t n + s$ . Then substituting into equation 11, we have,

$$\sum_{\tilde{a} \leq i < 2^T n} d_i + \mu_T - \sum_{1 \leq j < 2^T n} d_j + \sum_{2^T n < i \leq \tilde{s}} d_i = \sum_{\tilde{s} < j \leq \tilde{b}} d_j.$$

Which upon simplifying, we get

$$(12) \quad \sum_{\tilde{a} \leq i \leq \tilde{s}} d_i = \sum_{\tilde{s} < j \leq \tilde{b}} d_j + \sum_{1 \leq j < 2^t n} d_j.$$

We now have  $2^T n - 2^t n + 1 \leq \tilde{a} < \tilde{b} < 2^T n + 2^t n$ ,  $\tilde{s} \in [2^T n, \tilde{b}]$ . From Lemma 6, we have the following identities:

$$(13) \quad \sum_{1 \leq i < 2^t n} d_i = \sum_{2^T n + 2^t n + 1 \leq i < 2^T n + 2^{t+1}n} d_i$$

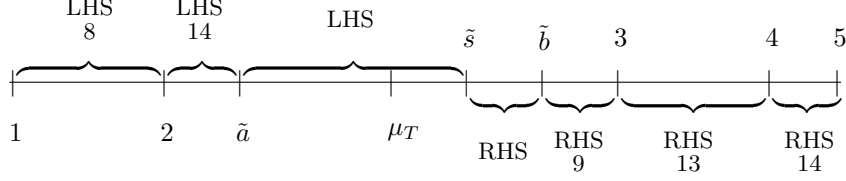
$$(14) \quad \sum_{2^T n - 2^t n \leq i < \tilde{a}} d_i = \sum_{2^T n + 2^{t+1}n \leq i < \tilde{a} + 2^{t+1}n + 2^t n} d_i$$

By substituting Equation 13 into Equation 12 and adding the RHS of Equation 9 and Equation 14 to the RHS of Equation 12 while adding the LHS of Equation 14 and Equation 8, we get the following

$$\sum_{\tilde{b}-2^{t+1}n < i < 2^T n - 2^t n} d_i + \sum_{2^T n - 2^t n \leq i < \tilde{a}} d_i + \sum_{\tilde{a} \leq i \leq \tilde{s}} d_i = \sum_{\tilde{s} < j \leq \tilde{b}} d_j +$$

$$(15) \quad \sum_{\tilde{b} < i < 2^T n + 2^t n} d_j + \sum_{2^T n + 2^t n + 1 \leq i < 2^T n + 2^{t+1} n} d_j + \sum_{2^T n + 2^{t+1} n \leq i < \tilde{a} + 2^{t+1} n + 2^t n} d_j.$$

We use the following picture to illustrate the substitutions:



where

- (1)  $\tilde{b} - 2^{t+1}n$
- (2)  $2^T n - 2^t n$
- (3)  $2^T n + 2^t n$
- (4)  $2^T n + 2^{t+1}n$
- (5)  $\tilde{a} + 2^{t+1}n + 2^t n$ .

Then by simplifying Equation 15, we have

$$\sum_{\tilde{b} - 2^{t+1}n < i \leq \tilde{s}} d_i = \sum_{\tilde{s} < j < \tilde{a} + 2^{t+1}n + 2^t n} d_j,$$

contradicting the definition of  $\mu_T$ .  $\square$

The last piece of the puzzle in order to show that the sum of the reciprocals of the sequences created using our construction converges is the following theorem.

**Theorem 8.** *If  $T > t + 2$  and  $\mu_T > \sum_{1 \leq i \leq 2^t n} d_i$ , then  $\mu_T > \sum_{1 \leq i < 2^{t+1} n} d_i$ .*

*Proof.* Assume by contradiction that  $\mu_T < \sum_{1 \leq i < 2^{t+1} n} d_i$ . Let  $\Delta = \sum_{1 \leq i \leq 2^{t+1} n} d_i - \mu_T > 0$ .

We now will show that  $\mu_{t+1} - \Delta = \mu_T - \sum_{1 \leq i < 2^{t+1} n} d_i$  satisfies 1 and 2 thus contra-

dicting the minimality of  $\mu_{t+1}$ . We now proceed by cases.

Case 1: Now suppose, by way of contradiction, that there exist  $1 \leq a < 2^{t+1}n < b < 2^{t+2}n$  and  $s \in [a, 2^t n]$  so that  $\sum_{a \leq i \leq s} d_i = \sum_{s < j < 2^{t+1}n} d_j + \mu_{t+1} - \Delta + \sum_{2^{t+1}n < j \leq b} d_j$ .

We then have

$$(16) \quad \sum_{a \leq i \leq s} d_i = \sum_{s < j < 2^{t+1}n} d_j + \mu_T - \sum_{1 \leq i < 2^{t+1}n} d_i + \sum_{2^{t+1}n < j \leq b} d_j.$$

Now let  $\tilde{a} = 2^T n - 2^{t+1}n + a$ ,  $\tilde{b} = 2^T n - 2^{t+1}n + b$  and  $\tilde{s} = 2^T n - 2^{t+1}n + s$ . Then substituting into equation 16, we have,

$$\sum_{\tilde{a} \leq i \leq \tilde{s}} d_i = \sum_{\tilde{s} < j < 2^T n} d_j + \mu_T - \sum_{1 \leq i < 2^{t+1}n} d_i + \sum_{2^T n < j \leq \tilde{b}} d_j.$$

Which upon simplifying, we get

$$(17) \quad \sum_{\tilde{a} \leq i \leq \tilde{s}} d_i + \sum_{1 \leq i < 2^{t+1}n} d_i = \sum_{\tilde{s} < j \leq \tilde{b}} d_j.$$

We now have  $2^T n - 2^{t+1}n + 1 \leq \tilde{a} < \tilde{b} < 2^T n + 2^{t+1}n$ ,  $\tilde{s} \in [\tilde{a}, 2^T n]$ . From Lemma 6, we have the following identities:

$$(18) \quad \sum_{1 \leq i < 2^{t+1}n} d_i = \sum_{2^T n - 2^{t+2}n + 1 \leq i < 2^T n - 2^{t+1}n} d_i$$

$$(19) \quad \sum_{2^T n - 2^{t+1}n \leq i < \tilde{a}} d_i = \sum_{2^T n + 2^{t+1}n \leq i < \tilde{a} + 2^{t+2}n} d_i$$

$$(20) \quad \sum_{\tilde{b} < i < 2^T n + 2^{t+1}n} d_i = \sum_{\tilde{b} - 2^{t+2}n - 2^{t+1}n < j < 2^T n + 2^{t+2}n} d_j$$

$$(21)$$

By substituting Equation 18 into equation 17 and adding the LHS of Equation 19 and Equation 20 to the LHS of equation 17 while adding the RHS of Equation 19 and Equation 20 to the RHS of equation 17, and we get the following

$$(22) \quad \begin{aligned} & \sum_{\tilde{a} \leq i \leq \tilde{s}} d_i + \sum_{2^T n + 2^{t+1}n < i < 2^T n - 2^t n} d_i + \sum_{2^T n - 2^t n \leq i < \tilde{a}} d_i + \sum_{\tilde{b} - 2^{t+1}n - 2^t n < i < 2^T n - 2^{t+1}n} d_i \\ & = \sum_{\tilde{s} < j \leq \tilde{b}} d_j + \sum_{2^T n + 2^t n \leq j < \tilde{a} + 2^{t+1}n} d_j + \sum_{\tilde{b} < j < 2^T n + 2^t n} d_j. \end{aligned}$$

Then by simplifying Equation 22, we have

$$\sum_{\tilde{b} - 2^{t+1}n - 2^t n < i \leq \tilde{s}} d_i = \sum_{\tilde{s} < j < \tilde{a} + 2^{t+1}n} d_j,$$

contradicting the definition of  $\mu_T$ .

Case 2: Because Case 1 is so similar to Case 1 of the previous theorem, Case 2 is also very similar. □

We now show that the sequence built from our construction converges.

**Corollary 9.** *Let  $\{m_k\}$  be a sequence constructed as above. Then  $\sum_{k \geq 1} \frac{1}{m_k}$  converges.*

*Proof.* By Theorems 7 and 8, we have

$$\mu_T > \sum_{1 \leq i \leq 2^{t+1}n} d_i.$$

It then follows that

$$\begin{aligned}
\mu_{T+3} &> \sum_{1 \leq i \leq 2^{T+1}n} d_i = \sum_{1 \leq i < 2^T n} d_i + \mu_T + \sum_{2^T n + 1 \leq i < 2^{T+1}n} d_i + \mu_{T+1} \\
&= \sum_{1 \leq i < 2^T n} d_i + \mu_T + \sum_{1 \leq i < 2^T n} d_i + \mu_{T+1} \\
&= 2 \sum_{1 \leq i < 2^T n} d_i + \mu_T + \mu_{T+1} \\
&> 3\mu_T
\end{aligned}$$

Then notice that by the same argument we have

$$\begin{aligned}
\mu_{T+4} &> 3\mu_{T+1} \\
\mu_{T+5} &> 3\mu_{T+2}.
\end{aligned}$$

Continuing in the same manner, we now have the following for  $k = 1, 2, 3, \dots$

$$\begin{aligned}
\mu_{T+3k} &> 3^k \mu_T \\
\mu_{T+3k+1} &> 3^k \mu_{T+1} \\
\mu_{T+3k+2} &> 3^k \mu_{T+2}.
\end{aligned}$$

Notice that

$$\begin{array}{rcl}
m_{2^{T+3+0}n+1} \text{ through } m_{2^{T+3+1}n} & \geq & 3\mu_T \\
m_{2^{T+3+1}n+1} \text{ through } m_{2^{T+3+2}n} & \geq & 3\mu_{T+1} \\
m_{2^{T+3+2}n+1} \text{ through } m_{2^{T+3+3}n} & \geq & 3\mu_{T+2} \\
\vdots & \vdots & \vdots \\
m_{2^{T+3+j}n+1} \text{ through } m_{2^{T+3+(j+1)}n} & \geq & 3^{\lfloor j/3 \rfloor + 1} \mu_{T+j \bmod 3}.
\end{array}$$

Therefore it follows that

$$\begin{aligned}
\sum_{k \geq 1} \frac{1}{m_k} &= \sum_{k \leq 2^{T+3}n} \frac{1}{m_k} + \sum_{j \geq 0} \left[ \sum_{2^{T+3+j}n < i \leq 2^{T+3+(j+1)}n} \frac{1}{m_j} \right] \\
&\leq \sum_{k \leq 2^{T+3}n} \frac{1}{m_k} + 8 \cdot 2^T \sum_{j=0,3,6,\dots} \frac{2^j}{3^j \mu_T} + 8 \cdot 2^T \sum_{j=1,4,7,\dots} \frac{2^j}{3^j \mu_{T+1}} + \\
&\quad 8 \cdot 2^T \sum_{j=2,5,8,\dots} \frac{2^j}{3^j \mu_{T+2}} \\
&\leq \sum_{k \leq 2^{T+3}n} \frac{1}{m_k} + \frac{8 \cdot 2^T}{\mu_T} \sum_{j=0,3,6,\dots} \frac{2^j}{3^j} + \frac{8 \cdot 2^T}{\mu_{T+1}} \sum_{j=1,4,7,\dots} \frac{2^j}{3^j} + \frac{8 \cdot 2^T}{\mu_{T+2}} \sum_{j=2,5,8,\dots} \frac{2^j}{3^j} \\
&< \infty.
\end{aligned}$$

□

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