CHAPTER 1 NOTES

TED LAI

Section 1.2 Real Numbers

Definition. A real number r is a *rational number* if

$$r = \frac{a}{b} \qquad \text{where } a \text{ and } b \text{ are integers and } b \neq 0.$$

Some examples of rational numbers are: $\frac{1}{2}$, $\frac{-21}{5}$, $3\frac{2}{7}$, 23.12, $0.\bar{3}$ ($=\frac{1}{3}$). Also note that each integer n is a rational number since $n = \frac{n}{1}$, which is a fraction of two integers.

Definition. A real number is an *irrational number* if it is not a rational number.

Some examples of irrational numbers are: $\sqrt{2}$, π, $0.110001\ldots$, $\frac{1+\sqrt{5}}{2}$ (the golden ratio).

Properties of Real Numbers. Let a, b, and c be real numbers. Then they satisfy the following properties:

Property 1 (The Commutative Property of Addition)**.**

$$a + b = b + a$$

Property 2 (The Commutative Property of Multiplication)**.**

$$ab = ba$$

Property 3 (The Associative Property of Addition)**.**

$$(a + b) + c = a + (b + c)$$

Property 4 (The Associative Property of Multiplication)**.**

$$(ab)c = a(bc)$$

Property 5 (The Distributive Property)**.**

$$a(b + c) = ab + ac \qquad \textit{and} \qquad (b + c)a = ba + ca$$

So, what's the point of these trivial properties? Let's illustrate their use by an example.

Example 1. *Let x, y, z, and w be real numbers. Show $(x+y)(z+w) = xz+yz+xw+yw$.*

$$\begin{aligned}
(x+y)(z+w) &= (x+y)z + (x+y)w && \textit{by Property 5} \\
&= (xz+yz) + (xw+yw) && \textit{by Property 5} \\
&= [(xz+yz) + xw] + yw && \textit{by Property 1} \\
&= xz + yz + xw + yw
\end{aligned}$$

Alternatively, one could verify the property numerically with explicit numbers, but it would take *forever* to check all possible combinations of real numbers.

Properties of Fractions. Let a, b, c, d be real numbers with $b \neq 0$ and $d \neq 0$. Then the following properties hold:

Property 6.

$$a = \frac{a}{1}$$

Property 7.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

Property 8.

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

Property 9.

$$\textit{If } c \neq 0, \textit{ then } \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}.$$

Sets and Intervals. Loosely speaking, a *set* is a collection of objects. The objects contained in a set are called *elements* of the set. A set could contain a finite number of elements or infinitely many elements.

Notation. Let S be a set. $a \in S$ denotes that a is an element of S. $a \notin S$ denotes that a is not an element of S.

Definition. Let S and A be sets. If every element contained in A is also contained in S, then A is a *subset* of S.

Sets frequently seen in class:

Notation	Description
$\mathbb{R}$	the set of all real numbers
$\mathbb{Q}$	the set of all rational numbers
$\mathbb{Z}$	the set of all integers
$\mathbb{R}^2$	the set of all ordered pairs of real numbers

Notation. Let S and A be sets. $A \subseteq S$ denotes that A is a subset of S.

We say two sets A and B are equal if $A \subseteq B$ and $B \subseteq A$.

Notation. $\{x \in S | \cdots\}$ denotes the set of all x in S such that ...

Invervals are subsets of $\mathbb{R}$ (see table above), and they correspond to line segments on the *real number line*. There are 9 types of intervals. Each one is represented in interval notation below.
Let a and b be real numbers such that $a \leq b$.

Notation	Set Description
(a, b)	$\{x \in \mathbb{R} \| a < x < b\}$
$[a, b]$	$\{x \in \mathbb{R} \| a \leq x \leq b\}$
$[a, b)$	$\{x \in \mathbb{R} \| a \leq x < b\}$
$(a, b]$	$\{x \in \mathbb{R} \| a < x \leq b\}$
(a, ∞)	$\{x \in \mathbb{R} \| x > a\}$
$[a, \infty)$	$\{x \in \mathbb{R} \| x \geq a\}$
$(-\infty, b)$	$\{x \in \mathbb{R} \| x < b\}$
$(-\infty, b]$	$\{x \in \mathbb{R} \| x \leq b\}$
$(-\infty, \infty)$	$\mathbb{R}$

(see pg 15 in textbook for graphs)

Section 1.3 Exponents and Radicals

Integer Exponents.

Notation. If a is a real number and n is a positive integer, then the *nth power* of a is

$$a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ times}}.$$

Observe that for any real number a and any positive integers m and n,

$$\begin{aligned} a^m a^n &= (\underbrace{a \times a \times \cdots \times a}_{m \text{ times}})(\underbrace{a \times a \times \cdots \times a}_{n \text{ times}}) \\ &= \underbrace{a \times a \times \cdots \times a}_{m+n \text{ times}} \qquad \text{by Property 4 in Section 1.2} \\ &= a^{m+n} \end{aligned}$$

Also one could show that $(a^m)^n = a^{mn}$, $(ab)^n = a^n b^b$ for any real number b, and $(\frac{a}{b})^n = \frac{a^n}{b^n}$ for any nonzero real number b.

Radicals.

Definition. Let a be a real number such that $a \geq 0$. A *square root* of a is a number b such that $b^2 = a$.

It can be shown that every positive number has exactly two distinct square roots: a positive one and a negative one. For example, 2 amd -2 are both square roots of 4 since $2^2 = 2 \times 2 = 4$ and $(-2)^2 = (-2) \times (-2) = 4$.

Notation. Let a be a real number such that $a \geq 0$. If a is positive, $\sqrt{a}$ denotes the positive square root of a. If $a = 0$, then $\sqrt{a}$=0.

Properties of Square Roots. Let a, b be real numbers with $a \geq 0$ and $b \geq 0$. Then the following properties hold:

Property 10.

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

Property 11.

$$\textit{If } b \neq 0, \textit{ then } \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$