RESEARCH STATEMENT

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Introduction

My primary research interests lie in extremal combinatorics, particularly intersection theorems in extremal set theory. A starting point in this line of research is the following question: Consider a collection of subsets of an n-element set X, such that no pair of subsets in the collection is disjoint. Call it an *intersecting* family. How large can such an intersecting family be? As it turns out, this question is surprisingly easy to answer. An intersecting family of subsets can have size at most 2^{n-1}, because for any subset A, at most one out of the pair $(A, X \setminus A)$ can be in the family. Furthermore, one of the structures which attains this *extremal* number is the family of subsets which contain a specific element, called a *star*.

A related question is the following: How large can an intersecting family of r-subsets of X be? If $r > n/2$, any pair of r-subsets have a non-empty intersection, but the case $r \leq n/2$ is non-trivial. In the paper that initiated the study of intersecting set systems, Erdös, Ko and Rado (1961) proved the following seminal result.

Theorem 1. *(Erdös-Ko-Rado) For a set $X = [n]$ and $2 \leq r \leq n/2$, if $\mathcal{A}$ is an intersecting family of r-subsets of X, then $|\mathcal{A}| \leq \binom{n-1}{r-1}$.*

A very pleasing fact about this extremal problem, which was shown later as part of a much stronger result by Hilton and Milner (1967), is that when $r < n/2$, the *star* is the only extremal structure.

The Erdös-Ko-Rado theorem is one of the fundamental theorems in combinatorics, and has inspired a large number of beautiful results, many of which have found applications not only within combinatorics, but also in the fields of information theory and probability. A particularly elegant application to probability was by Liggett (1977), who proved a result on sums of independent Bernoulli random variables using the bound in Theorem 1.

The broader area of extremal set theory also has connections to the theory of computing. For instance, the fundamental *lower bounds problem*, which is to prove that a given function cannot be computed within a certain amount of time or space, is an extremal problem, and techniques from extremal set theory have been extensively used to prove results of this type. A striking example was due to Razborov (1985), who used the *Sunflower Lemma* of Erdös and Rado (1960) to prove a lower bounds argument for monotone circuits.

Chvátal's Conjecture and Erdös-Ko-Rado Graphs

We say that the power set of a set X, denoted by 2^X, is *EKR*, since the set of maximum-sized intersecting subfamilies of 2^X contains a star. Similarly, we say that the family $X^{(r)}$, for $r < |X|/2$, is *strictly EKR*, since every member in the set of maximum-sized intersecting subfamilies is a star. 2^X is also a special example of a *hereditary* family, also referred to in the literature as an *ideal* or a *downset*. A family $\mathcal{F}$ is said to be hereditary if $A \in \mathcal{F}$ and $A' \subseteq A$ implies that $A' \in \mathcal{F}$. My dissertation research has primarily focused on generalizations of Theorem 1 motivated by the following long standing conjecture of Chvátal(1974).

Conjecture 2. *(Chvátal) If $\mathcal{F}$ is a hereditary family, then $\mathcal{F}$ is EKR.*

There are a few results which verify the conjecture for specific hereditary families. Among the most important ones is a result of Chvátal himself (1974). Let $\mathcal{F}$ be a hereditary family on a set X, which has a total ordering of its elements induced by a relation $\preceq$. Chvátal proved the conjecture when $\mathcal{F}$ is *compressed*, i.e. if $\{x_1, \ldots, x_r\} \in \mathcal{F}$ and $y_i \preceq x_i$ for each $1 \leq i \leq r$, then $\{y_1, \ldots, y_r\} \in \mathcal{F}$. Snevily (1992) further extended Chvátal's theorem and proved the conjecture when the family is compressed with respect to a specific element x, i.e. if $F \in \mathcal{F}$ such that $y \in F$ but $x \notin F$, then $F \setminus \{y\} \cup \{x\} \in \mathcal{F}$.

One of the recent generalizations of Theorem 1 considers hereditary families of vertex sets of a graph G. It is not hard to observe that the family of all *independent* vertex sets (subsets of vertices containing no edges) of a graph G is hereditary. Holroyd, Spencer and Talbot (2005) consider *uniform* subfamilies of this family, i.e. subfamilies containing all subsets of the same size. For a graph G, vertex $v \in V(G)$ and some integer $r \geq 1$, denote the family of independent sets of size r of $V(G)$ by $\mathcal{I}^{(r)}(G)$ and the star subfamily $\{A \in \mathcal{I}^{(r)}(G) : v \in A\}$ by $\mathcal{I}_v^{(r)}(G)$. Call G (strictly) r-EKR if $\mathcal{I}^{(r)}(G)$ is (strictly) EKR.

Earlier results by Berge (1974), Deza and Frankl (1983), and Bollobas and Leader (1997), while not explicitly stated in graph-theoretic terms, hint in this direction. The following interesting conjecture was posed by Holroyd and Talbot (2005). For graph G, let $\mu(G)$ be the minimum size of a maximal independent set.

Conjecture 3. *Let G be any graph and let $1 \leq r \leq \frac{1}{2}\mu$; then G is r-EKR(and is strictly so if $2 < r < \frac{1}{2}\mu$).*

Borg and Holroyd (2008) proved this conjecture for certain classes of graphs, which have an isolated vertex as a component. One of the primary contributions of my dissertation has been to extend the techniques developed in these papers and prove a more general theorem for a large class of graphs, that encompasses results by Borg-Holroyd(2008) and Holroyd et al(2005). Call a graph *chordal* if every cycle of length at least 4 has a chord, i.e. an edge between non-adjacent vertices of the cycle.

Theorem 4. *(Hurlbert, Kamat) If G is a disjoint union of chordal graphs, including at least one isolated vertex, and if $r \leq \frac{1}{2}\mu(G)$, then G is r-EKR.*

The isolated vertex condition, in the hypothesis of the theorem, allows us to determine the center of a maximum star in the graph (in a graph with an isolated vertex, it is not hard to show that one of the maximum stars is centered at the isolated vertex). More importantly, it makes it easy to extend Theorem 4 in the direction of Chvátal's conjecture. Let $\mathcal{I}^{(\leq r)}(G)$ be the hereditary family of all independent vertex sets of size at most r.

Corollary 5. *If G is a disjoint union of chordal graphs, including at least one isolated vertex, and if $r \leq \frac{1}{2}\mu(G)$, then $\mathcal{I}^{(\leq r)}(G)$ satisfies Conjecture 2.*

Future Research Directions

Towards Chvátal's conjecture

One of the main motivations behind the graph-theoretic generalization of Erdös-Ko-Rado is to be able to construct large classes of hereditary families with special structure. It seems possible that the techniques we've developed can be extended to prove Conjecture 3 for larger classes of graphs containing an isolated vertex. In the inductive proof of Theorem 4, the *simplicial* structure of a chordal graph helps us use a powerful shifting technique. Classes of graphs with a similar, but more general structure, such as *vertex decomposable* graphs, could also be tackled by devising appropriate modifications of these techniques.

It would also be desirable if the isolated vertex condition in Theorem 4 could be removed, although this seems harder. It is one of the central difficulties involved in proving similar results for graphs without isolated vertices. A few interesting questions emerge, two of which are as follows: is it possible to construct large classes of graphs without isolated vertices which satisfy Conjecture 3? Also, given a graph G, where is a maximum star centered at? We've achieved some success in answering these questions. We construct special classes of chordal graphs, by blowing up edges of a path into complete graphs, and prove that for any $r \geq 1$, these graphs are r-EKR. In other words, we prove a statement much stronger than Conjecture 3 for these graphs.

We also consider the problem of finding centers of maximum stars in trees, and make the following conjecture.

Conjecture 6. *Let T be a tree on n vertices, and let $r \geq 1$. Then, there exists a leaf vertex l (vertex of degree 1) such that for any $x \in V(T)$, $|\mathcal{I}_x^r(T)| \leq |\mathcal{I}_l^r(T)|$.*

This conjecture makes heuristic sense, because taking a vertex with large degree as the center of the star prevents a large number of vertices from appearing in any set of the star. We give a proof of the conjecture for small values of r. However, the general case seems hard to prove (or disprove). One of the reasons is that a stronger form of the conjecture is not true. There exist trees where some leaf centered stars are smaller than non-leaf centered ones. In more general terms, it is not necessarily true that vertices with smaller degrees have larger stars centered at them. This *degree sort* property also fails to hold within vertices of a single partite set.

We also propose the following weakening of Chvátal's conjecture, for all graphs.

Conjecture 7. *If a graph G satisfies Conjecture 3, then $\mathcal{I}^{(\leq r)}(G)$ is EKR for all $r \leq \mu(G)/2$.*

As discussed in the earlier section, for graphs with isolated vertices, this conjecture is easy to verify because for any value of r, we can always fix the isolated vertex as a maximum star center. The conjecture remains open for graphs without isolated vertices, because it is possible that for different values of r, the maximum stars are centered at different vertices.

Stability for Intersection Theorems

The classical extremal problem is to determine the maximum size, and possibly structure, of a set system on a given ground set of size n, which avoids a given forbidden configuration $\mathcal{F}$. For example, the Erdös-Ko-Rado theorem finds the maximum size and structure of a set system on the set $X = [n]$, which does not have a pair of disjoint subsets. Often, only a few trivial structures attain this extremal number. A natural further step is to ask whether non-extremal families, which have size close to the extremal number, also have structure similar to any of the extremal structures. This approach was first pioneered by Simonovits (1966), to answer a question in extremal graph theory, and a similar notion for set systems was recently formulated by Mubayi (2007). Stability theorems are interesting for a couple of reasons. They prove the *continuity* of the structures close to the extremal number and consequently, shed more light on the problem under consideration than just the extremal result. Somewhat surprisingly, stability results have also been used to prove exact results, and this application is particularly valuable in extremal hypergraph theory, where exact results are typically harder to prove. One of the first results of this kind in extremal set theory was the theorem of Hilton and Milner (1967) which, by giving an upper bound on the maximum size of non-star intersecting families, proved a strong stability result for the Erdös-Ko-Rado theorem.

One of the goals of my dissertation research, and postdoctoral career, is to formulate and prove such stability results for graph-theoretic variants of the Erdös-Ko-Rado theorem.

Cross-intersecting Families

A set of families $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$, for some $k \geq 2$, is cross-intersecting if $A_i \in \mathcal{A}_i$ and $A_j \in \mathcal{A}_j$ implies $A_i \cap A_i \neq \emptyset$ for any $i \neq j$. Hilton (1977) proved a nice generalization of Erdös-Ko-Rado, by giving an upper bound on $\sum_{i=1}^{k} |\mathcal{A}_i|$. We recently discovered the following extension of Hilton's theorem.

Theorem 8. *(Kamat) Let $G_1, \ldots, G_n$ be n complete graphs, each of size at least 2. Let m_i be the size of G_i, for each $1 \leq i \leq n$. Let G be the disjoint union of these n graphs, and let $r \leq n$. Also, let x be such that $\mathcal{I}_x^{(r)}(G)$ is maximum. For some $k \geq 2$, let $\mathcal{A}_1, \ldots, \mathcal{A}_k \subseteq \mathcal{I}^r(G)$ be cross-intersecting families. Then,*

$$\sum_{i=1}^{k} |\mathcal{A}_i| \leq \begin{cases} |\mathcal{I}^{(r)}(G)| & \text{if } k < min_{i=1}^{n}\{|G_i|\} \\ k|\mathcal{I}_x^{(r)}(G)| & \text{if } k \geq min_{i=1}^{n}\{|G_i|\} \end{cases}$$

If equality holds for the case $k < min_{i=1}^{n}\{|G_i|\}$, then $\mathcal{A}_1 = \mathcal{I}^{(r)}(G)$ and $\mathcal{A}_i = \emptyset$ for each $2 \leq i \leq n$.

It seems like the bounds in Theorem 8 could be potentially useful for proving that the Erdös-Ko-Rado property of certain disjoint unions of complete graphs is stable. This is one of the motivations for developing a graph-theoretic generalization of Hilton's theorem, along the lines of Theorem 8, and it is another research direction I'm interested in pursuing.