

**Problem 1**

Find the volume of the largest box that will fit inside the ellipsoid

$$\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{1} = 1$$

Note that the volume of the box is eight times that portion of the box that is in the first octant. So if  $(x, y, z)$  is where the box intersects the ellipsoid in the first octant, then setting  $V = 8xyz$  and absorbing the constraint into the function, it suffices to maximize

$$f(x, y) = 8xy\sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$$

(Note: only solutions with  $x > 0$ ,  $y > 0$ , and  $z > 0$  are from the first octant.)

**Sample problem**

Find the area of the largest rectangle in the first quadrant that will fit inside the curve

$$\frac{x^p}{a^p} + \frac{y^p}{b^p} = 1$$

So if  $(x, y)$  is where the rectangle intersects the curve in the first quadrant, then we want to maximize  $f(x, y) = xy$  subject to the constraint  $\frac{x^p}{a^p} + \frac{y^p}{b^p} = 1$ .

**Solution**

Using Lagrange multipliers, our three equations become

$$(1) \quad y = \lambda p \frac{x^{p-1}}{a^p} \implies xy = \lambda p \frac{x^p}{a^p}$$

$$(2) \quad x = \lambda p \frac{y^{p-1}}{b^p} \implies xy = \lambda p \frac{y^p}{b^p}$$

$$(3) \quad \frac{x^p}{a^p} + \frac{y^p}{b^p} = 1$$

If we sum the first two equations, we get  $2xy = \lambda p \left( \frac{x^p}{a^p} + \frac{y^p}{b^p} \right)$  and using the third equation we get  $2xy = p\lambda$  or  $\lambda = 2xy/p$ . Putting this back into the first two equations, eqn one becomes

$$y = \lambda p \frac{x^{p-1}}{a^p} \implies y = 2 \frac{x^p}{a^p} y$$

which implies either  $y = 0$  (min area for a rectangle) or  $x = \frac{a}{\sqrt[p]{2}}$ .

And equation 2 becomes

$$x = \lambda p \frac{y^{p-1}}{b^p} \implies x = 2 \frac{y^p}{b^p} x$$

which implies either  $x = 0$  (min area for a rectangle) or  $y = \frac{b}{\sqrt[p]{2}}$ .

**Problem 2**

Use the above results to find the dimensions of the largest rectangle in the first quadrant under the two curves below. Include a sketch of the curve and the rectangle.

a) The line  $2x + 3y = 6$

b) The ellipse  $\frac{x^2}{9} + \frac{y^2}{4} = 1$

**Problem 3**

As in the previous sample problem, use Lagrange multipliers to find the volume of the largest box in the first octant that will fit inside the surface

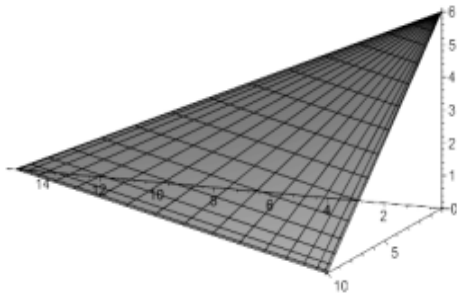
$$\frac{x^p}{a^p} + \frac{y^p}{b^p} + \frac{z^p}{c^p} = 1$$

Note that if  $(x, y, z)$  is where the box intersects the surface in the first octant, then we want to maximize  $f(x, y, z) = xyz$  subject to the constraint  $\frac{x^p}{a^p} + \frac{y^p}{b^p} + \frac{z^p}{c^p} = 1$ .

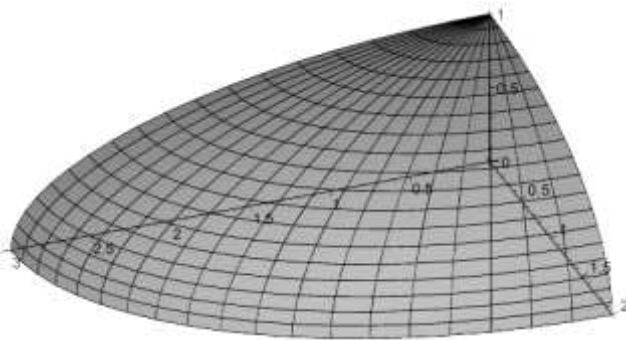
**Problem 4**

Use your results from Problem 3 to find the largest box in the first octant that will fit inside the three surfaces:

- a) The plane  $2x + 3y + 5z = 30$



- b) The ellipsoid  $\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{1} = 1$



- c) The surface

$$\frac{\sqrt{x}}{3} + \frac{\sqrt{y}}{\sqrt{5}} + \frac{\sqrt{z}}{\sqrt{6}} = 1$$

