Find the inverse of $f(x) = \frac{x}{3x - 9}$. Also find the domain and range of $f$ and $f^{-1}$.

We have not yet determined whether $f$ has an inverse, but certainly it does if we restrict its domain enough. If we must restrict the domain, perhaps there is a natural way to do it without throwing away too much. For now, assume the largest possible domain $D_0$ for $f$,

$$D_0 = \mathbb{R} - \{3\}.$$ 

The procedure for finding the inverse of $f$ is to interchange $x$ and $y$ in the formula for $f$ and solve for $y$. We have

$$\begin{align*}
x &= \frac{y}{3y - 9} \\
3xy - 9x &= y \\
(3x - 1)y &= 9x \\
y &= \frac{9x}{3x - 1}
\end{align*}$$

Let $g(x) = \frac{9x}{3x - 1}$. Think of this as the formulaic inverse of $f$. We still are not sure whether $g$ is the function inverse of $f$. In order for $g$ to be the function inverse of $f$, the following must hold:

1. $f$ is one-to-one
2. if $f$ has domain $D$ and range $R$, then $g$ has domain $R$ and range $D$
3. for all $x \in D$, $g(f(x)) = x$
4. for all $x \in R$, $f(g(x)) = x$.

What is the range of $f$? If $x$ is any real number but 1/3, then

$$f(g(x)) = \frac{9x/(3x - 1)}{3[9x/(3x - 1)] - 9} = \frac{9x}{27x - 27x + 9} = x.$$ 

That is, $x \in \text{ran}(f)$. So $\text{ran}(f)$ contains every real number except possibly 1/3. Can $1/3$ be in $\text{ran}(f)$? If so, then there is a solution to $1/3 = f(x) = x/(3x - 9)$.
Then $1 = \frac{x}{x-3}$ and $x - 3 = x$. There is no such solution. So $1/3 \notin \text{ran}(f)$, and the range $R_0$ of $f$ corresponding to domain $D_0$ is $R_0 = \mathbb{R} - \{1/3\}$.

Is $f$ one-to-one? Let $x_1, x_2 \in D_0$ with $f(x_1) = f(x_2)$. Then $f(x_1) \neq 1/3$ and $f(x_2) \neq 1/3$. So $f(x_1), f(x_2) \in \text{dom}(g)$. Let $x$ be any number but 3. Then

$$g(f(x)) = \frac{9[x/(3x-9)]}{3[x/(3x-9)] - 1} = \frac{9x}{3x - 3x + 9} = x.$$ 

Since $x_1, x_2 \in \text{dom}(f)$, we have $x_1 \neq 3$ and $x_2 \neq 3$. So $g(f(x_1)) = x_1$, $g(f(x_2)) = x_2$, and $x_1 = x_2$. This shows that the only way to have $f(x_1) = f(x_2)$ is if $x_1 = x_2$. In other words, we have shown that $f$ passes the horizontal line test. Have we verified all the conditions for $g$ to be the inverse of $f$?