Approximating \( \sin(1) \) using sum identities

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For certain angles \( \alpha \) it is easy to find the value of \( \sin(\alpha) \). Easiest is when \( \alpha \) is an integer multiple of \( \pi/2 \). For instance, if \( n \) is an integer, then we could write\(^1\)

\[
\sin(n\pi/2) = \begin{cases} 
0 & \text{if } 4\mid(n - 0) \\
1 & \text{if } 4\mid(n - 1) \\
0 & \text{if } 4\mid(n - 2) \\
-1 & \text{if } 4\mid(n - 3).
\end{cases}
\]

We also know the value of \( \sin(\alpha) \) for angles \( \alpha \) that are integer multiples of \( \pi/4 \) but not of \( \pi/2 \), such as \(-3\pi/4\). For such angles we determine the value of \( \sin(\alpha) \) using a triangle with internal angles \( \pi/4, \pi/4, \pi/2 \). In this case \( \sin(\alpha) = \pm 1/\sqrt{2} \).

Finally, if \( \alpha \) is an integer multiple of \( \pi/6 \) but not of \( \pi/2 \), then we may determine the value of \( \sin(\alpha) \) using a triangle with internal angles \( \pi/6, \pi/3, \pi/2 \). The value is \( \pm 1/2 \) or \( \pm \sqrt{3}/2 \).

However, for most angles \( \alpha \) the value of \( \sin(\alpha) \) is not so easy to find. For instance, 1 is not an integer multiple of \( \pi/6 \) or \( \pi/4 \). Since \( \pi/3 \approx 1.047 \) is not far from 1, a first approximation might be\(^2\)

\[
\sin(1) \approx \sin(\pi/3) = \sqrt{3}/2 \approx .86603.
\]

We can do better using sum, difference, and half angle identities.

\[
\sin(\alpha \pm \beta) = \sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta) \tag{1}
\]

\[
\cos(\alpha \pm \beta) = \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta) \tag{2}
\]

\[
\sin(\alpha/2) = \pm \sqrt{\frac{1 - \cos \alpha}{2}} \tag{3}
\]

\[
\cos(\alpha/2) = \pm \sqrt{\frac{1 + \cos \alpha}{2}} \tag{4}
\]

\(^1\)Recall that for integers \( k \) and \( n \), \( k\mid n \) means that \( k \) divides \( n \) evenly.

\(^2\)There are other formulas, like \( \sin(n\pi/2) = (\lfloor -1 \rfloor^{n/2} - (-1)^{\lfloor n/2 \rfloor})/2 \).

\(^3\)We rely on the fact that \( \sin \) is a continuous function.
We compensate for the difference between 1 and \( \pi/3 \).

\[
1 \approx \pi/3 - .0472 \quad (5)
\]

\[
\approx \pi/3 - \pi/67 \quad (6)
\]

\[
\approx \pi/3 - \pi/64 \quad (7)
\]

\[
= \frac{\pi}{3} - \frac{1}{2} \left( \frac{1}{2} \left( \frac{1}{2} \left( \frac{\pi}{4} \right) \right) \right) \quad (8)
\]

\[
\sin(1) \approx \sin \left( \frac{\pi}{3} - \frac{1}{2} \left( \frac{1}{2} \left( \frac{1}{2} \left( \frac{\pi}{4} \right) \right) \right) \right) \quad (9)
\]

The last expression can be evaluated using the identities above.\(^4\)

The study of calculus yields a more general tool for approximation, the Taylor polynomial. In this worksheet you will compare approximations obtained with trigonometric identities to those obtained with Taylor polynomials.

Complete the following, always providing five figures after the decimal point for numeric answers.

1. Use trigonometric identities to estimate \( \sin(1) \).

   (a) Use half angle identities to write an expression for \( \cos(\pi/32) \) that involves only adding, subtracting, multiplying, dividing, and taking square roots\(^5\) of real numbers (no trigonometric functions). Then evaluate the expression.\(^6\)

   (b) Use the approximation from (9), a difference identity, half angle identities, and your answer from part (a) to approximate \( \sin(1) \).

2. Use Taylor polynomials to estimate \( \sin(1) \). Evaluate \( T_n \sin(x) \) at \( x = 1 \) for \( n = 3, 5, 11 \).

   (a) \( T_3 \sin(x) = x - x^3/6 \)

   (b) \( T_5 \sin(x) = x - x^3/6 + x^5/120 \)

   (c) \( T_{11} \sin(x) = x - x^3/6 + x^5/120 - x^7/5040 + x^9/362880 - x^{11}/39916800 \)

3. Bonus: Use a protractor and ruler to estimate \( \sin(1) \).

4. Compare your approximations by listing them in order of increasing distance from the value of \( \sin(1) \) obtained using the SIN button on your calculator.

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\(^4\) Angles of radian measure \( \pi, \pi/2, \pi/4, \pi/6, \ldots \) can be used to approximate the value of \( \sin \) to arbitrary precision for any angle in \([0, 2\pi]\). For instance, to find \( \sin(2.5) \), first convert radians to revolutions: 2.5 radians \( \approx .39789 \) revolutions. Now convert this number to binary: .39789\(_{10} \approx .01100112\). From the binary expansion we get 2.5 \( \approx \pi/2 + \pi/4 + \pi/32 + \pi/64 \), and now the approximate value of \( \sin(2.5) \) can be evaluated using half angle and sum identities.

Challenge: Program your calculator or a computer to evaluate \( \sin \) this way.

\(^5\) These operations are associated with simple algorithms which can be executed with pencil and paper or a cheap microprocessor. For finding square roots, consider Newton’s iteration: http://mathworld.wolfram.com/SquareRoot.html.

\(^6\) Tip: Enter the function \( Y = \sqrt{1+X^2} \) into your calculator.