Solving equations like $2 \sin x + \sqrt{3} = 0$

MAT 170 – David Smith

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We wish to find all solutions\(^1\) to the equation:

$$\sin x = -\frac{\sqrt{3}}{2} \tag{1}$$

Of course this means finding all real numbers $x$ which make the equation true. Since $\sin$ is periodic, it suffices to find solutions in a “large enough” restricted domain, then “extend” the solutions to the rest of the real number line. Since $\sin$ has period $2\pi$, a reasonable restricted domain is the interval $[0, 2\pi]$. Other possible choices are $[-\pi, \pi]$ or $(-7.77, -7.77 + 2\pi)$. The domain should be an interval of length equal to the period of the function in question ($2\pi$ in this example), with the interval containing at least one of its endpoints.

The solutions of (1) on the domain $[0, 2\pi]$ are:

$$x = \frac{4\pi}{3}, \frac{5\pi}{3} \tag{2}$$

These solutions are obtained by recognizing $\frac{\sqrt{3}}{2}$ to be the length of a leg of one of our “special right triangles”, the one with internal angles $\frac{\pi}{6}, \frac{\pi}{3},$ and $\frac{\pi}{2}$. Note that these two solutions correspond to angles with terminal rays in quadrants III and IV, respectively.

Now we “extend” the solutions in (2). Since $4\pi/3$ is a solution of (1), all of

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\(^1\) (rather than solutions on a restricted domain such as $[0, 2\pi]$)
the following must be solutions of (1) (by the periodicity of sin):

\[
\begin{align*}
4\pi/3 & \quad - \quad 6\pi \\
4\pi/3 & \quad - \quad 4\pi \\
4\pi/3 & \quad - \quad 2\pi \\
4\pi/3 & \\
4\pi/3 & \quad + \quad 2\pi \\
4\pi/3 & \quad + \quad 4\pi \\
4\pi/3 & \quad + \quad 6\pi \\
\vdots 
\end{align*}
\]

Mathematical shorthand for this collection is \( \frac{4\pi}{3} + 2k\pi \), where \( k \) is any integer. We conclude that all solutions to (1) are given by \( \frac{4\pi}{3} + 2k\pi \) and \( \frac{5\pi}{3} + 2k\pi \), where \( k \) is any integer.