Today we studied a problem similar to the following: "There are 19 pink candy hearts, 3 blue candy hearts, and 10 red candy hearts in a bowl. Choose 9 hearts from the bowl. What’s the probability that exactly 3 are pink?"

Here’s one way. Let $A$ be the alphabet

$$\{p_1, p_2, \ldots, p_{19}\} \cup \{b_1, b_2, b_3\} \cup \{r_1, r_2, \ldots, r_{10}\}.$$ 

Define the experiment so that an outcome is a word of length 9 from alphabet $A$ (without repeated letters). So there are $P(19+3+10, 9)$ possible outcomes. We enumerate the favorable outcomes as follows:

1. choose 3 positions of the 9 for pink hearts to occupy: $C(9, 3)$
2. choose pink hearts for those 3 positions: $P(19, 3)$
3. choose nonpink hearts for the remaining 6 positions: $P(3 + 10, 6)$.

The answer is

$$\frac{C(9, 3)P(19, 3)P(13, 6)}{P(32, 9)} = \frac{9!19!13!23!}{6!3!16!7!32!}.$$ 

Here’s another way. Define the experiment so that an outcome is a subset of $A$ of size 9. So there are $C(19 + 3 + 10, 9)$ possible outcomes. Given any outcome, we can separate the 9 hearts into a set $P$ containing the pink hearts and a set $N$ containing the nonpink hearts. The favorable outcomes have $|P| = 3$ and $|N| = 6$, and we enumerate them:

1. choose $P$: $C(19, 3)$
2. choose $N$: $C(3 + 10, 6)$.

The answer is

$$\frac{C(19, 3)C(13, 6)}{C(32, 9)} = \frac{19!13!23!9!}{16!3!7!6!32!1!}.$$ 

same as above. Note that if you obtained an answer in this form on your homework, you gave each heart an identity, either explicitly or implicitly, in order to find $C(19, 3)$ different ways to pick a set of 3 white hearts (rather than just 1 way – see the definition of a combination).

Two different approaches yield the same answer. Is this reassuring or disturbing? Here’s the most helpful explanation I can think of. Many of you seem to find the second method easy to accept: since there is no order in a
handful of hearts, you use combinations and a simple two-step process for constructing the 3-pink 9-heart handfuls.

But this problem could equally be encountered by someone for whom drawing a “you seem nice” heart before an “i think you’re special” heart is quite different from the other way around. She discerns two hearts of the same color to be distinct, and she keeps them lined up in the order in which she drew them. Wouldn’t you expect the probability that her collection of 9 hearts has exactly 3 pink ones to be the same as yours? The fact that order matters to her doesn’t change her luck. She observes that there are \( P(32, 9) \) possible outcomes, far more than your mere \( C(32, 9) \) outcomes, since she cares about something you don’t. If she follows your two-step scheme for enumerating 3-pink 9-heart collections, she counts the number of ways to receive 3 pink hearts: \( P(19, 3) \), and the number of ways to receive 6 nonpink hearts: \( P(13, 6) \). Now the pink hearts are selected and ordered, as are the nonpink hearts. But she cares also how the pink hearts are ordered with respect to the nonpink hearts. So she takes into account which 3 of the 9 draws resulted in pink hearts: \( C(9, 3) \), giving the answer we obtained using the first method. Her denominator is larger than yours by a factor of \( 9! = 362880 \), while if she incorrectly used your two-step process, her numerator would be larger only by a factor of \( 3!6! = 6 \cdot 720 = 4320 \). By correctly adding the third step, she further increases her numerator by a factor of \( C(9, 3) = \frac{9!}{3!6!} = 84 \), and since \( 84 \cdot 4320 = 362880 \), we see that her ratio is the same as yours.