Exercise #1. Let \( f : (0, \infty) \to (0, 1) \) defined by \( f(x) = \frac{x}{x+1} \). Show that \( f(x) \) is a bijection. Find \( f^{-1} \) and give a proof.

**Proof.** First we’ll show \( f(x) \) is a bijection.

(1) We first show \( f \) is injective.

For any \( x_1, x_2 \in (0, \infty) \), suppose \( f(x_1) = f(x_2) \)

\[
f(x_1) = f(x_2) \Rightarrow \frac{x_1}{x_1 + 1} = \frac{x_2}{x_2 + 1}
\]

\[
\Rightarrow \frac{x_1}{x_1 + 1} \cdot (x_1 + 1)(x_2 + 1) = \frac{x_2}{x_2 + 1} \cdot (x_1 + 1)(x_2 + 1)
\]

\[
\Rightarrow (x_2 + 1)x_1 = (x_1 + 1)x_2
\]

\[
\Rightarrow x_2x_1 + x_1 = x_1x_2 + x_2
\]

\[
\Rightarrow x_1 = x_2
\]

Therefore \( f \) is injective.

(2) We then show \( f \) is surjective.

Let \( y \in (0, 1) \) be arbitrary.

Choose \( x = \frac{y}{1-y} \).

Since \( y > 0 \) and \( y < 1 \), then \( y > 0 \) and \( 1 - y > 0 \). Hence \( x = \frac{y}{1-y} > 0 \).

Thus \( x \in (0, \infty) \).

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Also we have
\[ f(x) = f\left(\frac{y}{1-y}\right) \]
\[ = \frac{y}{1-y} \]
\[ = \frac{y}{y + (1 - y)} \quad \text{multiplied by } 1 - y \text{ in both top and bottom.} \]
\[ = y \]
Therefore \( f \) is surjective.

Since \( f \) is both injective and surjective, \( f \) is bijective.

Since \( f \) is bijective, \( f^{-1} \) exists.
We claim \( f^{-1} : (0, 1) \to (0, \infty) \) is defined by
\[ f^{-1}(y) = \frac{y}{1-y}. \]

Proof of the claim: Let \( p(y) = \frac{y}{1-y} \) for any \( y \in (0, 1) \).
Then for any \( x \in (0, \infty) \),
\[ (p \circ f)(x) = p(f(x)) \]
\[ = \frac{f(x)}{1 - f(x)} \]
\[ = \frac{x/(x+1)}{1 - x/(x+1)} \]
\[ = \frac{x}{(x+1) - x} \quad \text{multiplied by } x+1 \text{ in both top and bottom.} \]
\[ = x \]
Also for any \( y \in (0, 1) \)
\[ (f \circ p)(y) = f(p(y)) \]
\[ = \frac{p(y)}{p(y) + 1} \]
\[ = \frac{y/(1-y)}{y/(1-y) + 1} \]
\[ = \frac{y}{y + (1 - y)} \quad \text{multiplied by } 1 - y \text{ in both top and bottom.} \]
\[ = y \]
Therefore, \( p = f^{-1} \). \qed
Exercise #2. Let $a, b \in \mathbb{R}$ with $a < b$. Define $g : (a, b) \to (0, 1)$ by

$$g(x) = \frac{x - a}{b - a}$$

(i) Prove that $g$ is a bijection. Find $g^{-1}$ and give a proof.
(ii) Use $f^{-1}$ in exercise #1 and $g$ to find a bijection $h$ from $(a, b)$ to $(0, \infty)$.
(iii) Find $h^{-1}$ directly and compare it with $g^{-1} \circ f$.

(i) Proof. First we show $g : (a, b) \to (0, 1)$ is a bijection.

(1) We first prove $g$ is injective.
For any $x_1, x_2 \in (a, b)$, suppose $g(x_1) = g(x_2)$,

$$g(x_1) = g(x_2) \Rightarrow \frac{x_1 - a}{b - a} = \frac{x_2 - a}{b - a}$$

$$\Rightarrow \frac{x_1 - a}{b - a} \cdot (b - a) = \frac{x_2 - a}{b - a} \cdot (b - a)$$

$$\Rightarrow x_1 - a = x_2 - a \quad \text{since} \quad b > a$$

$$\Rightarrow x_1 = x_2$$

Therefore $g$ is injective.

(2) We then show $g$ is surjective.
For any $y \in (0, 1)$. Let $x = (b - a)y + a$.
Since $0 < y < 1$ and $b > a$, $0 < (b - a)y < (b - a)$, hence $a < (b - a)y + a < b$.

$$g(x) = g((b - a)y + a)$$

$$= \frac{(b - a)y + a - a}{b - a}$$

$$= \frac{(b - a)y}{b - a}$$

$$= y$$

Therefore $g$ is surjective.
Therefore, $g$ is bijective. Since $g$ is bijective, $g^{-1}$ exists.
We claim that $g^{-1} : (0, 1) \to (a, b)$ is defined by

$$g^{-1}(y) = (b - a)y + a.$$
proof of the claim: Let \( p(y) = (b - a)y + a \) for any \( y \in (0, 1) \). Then for any \( x \in (a, b) \),

\[
(p \circ g)(x) = p(g(x)) \\
= (b - a)g(x) + a \\
= (b - a) \cdot \frac{x - a}{b - a} + a \\
= x - a + a \\
= x
\]

Also for any \( y \in (0, 1) \)

\[
(g \circ p)(y) = g(p(y)) \\
= \frac{p(y) - a}{b - a} \\
= \frac{(b - a)y + a - a}{b - a} \\
= \frac{(b - a)y}{b - a} \\
= y
\]

Therefore \( p = g^{-1} \). \qed

(ii) By theorem 5.5.6, since \( f : (0, \infty) \rightarrow (0, 1) \) is a bijection, \( f^{-1} : (0, 1) \rightarrow (0, \infty) \) exists and is also a bijection. Also we know that \( g : (a, b) \rightarrow (0, 1) \) is a bijection. By theorem 5.5.7, we can compose \( f^{-1} \) with \( g \) such that \( f^{-1} \circ g : (a, b) \rightarrow (0, \infty) \) is also a bijection. The bijection \( h(x) : (a, b) \rightarrow (0, \infty) \) is thus defined by

\[
h(x) = (f^{-1} \circ g)(x) \\
= f^{-1}(g(x)) \\
= \frac{g(x)}{1 - g(x)} \\
= \frac{x - a}{b - a} \\
= \frac{x - a}{b - x}
\]

(iii) Since \( g, f \) are bijections, by theorem 5.5.6, \( g^{-1} \) is a bijection. Then by theorem 5.5.7 \( g^{-1} \circ f : (0, \infty) \rightarrow (a, b) \) is also a bijection. For
any $x \in (0, \infty)$,
\[
(g^{-1} \circ f)(x) = g^{-1}(f(x)) \\
= (b - a)f(x) + a \\
= (b - a) \cdot \frac{x}{x + 1} + a \\
= \frac{(b - a)x + a(x + 1)}{x + 1} \\
= \frac{bx + a}{x + 1}
\]
Since $h$ is bijective, by theorem 5.5.6 $h^{-1}$ exists and by directly switching $x$ and $y$ in $h$ and solve for $y$, we obtain that
\[
h^{-1}(x) = \frac{bx + a}{x + 1}
\]
Therefore $g^{-1} \circ f$ and $h^{-1}$ are the same.
Also, by theorem 5.5.7, since $g$ and $f^{-1}$ are both bijective, $h^{-1} = (f^{-1} \circ g)^{-1} = g^{-1} \circ (f^{-1})^{-1} = g^{-1} \circ f$.

**Exercise #3.** Prove or disprove: $\mathbb{N} \setminus \{4, 5\}$ is an infinite set.

*Proof.* Let $A = \mathbb{N} \setminus \{4, 5\}$.
Define a function $f : A \to A$ by
\[
f(x) = x + 5
\]
for any $x \in A$. Next we’ll show that $f$ is a one-to-one correspondence, i.e., injective but not surjective.
For any $x_1, x_2 \in A$, suppose $f(x_1) = f(x_2)$, then $x_1 + 5 = x_2 + 5$, so $x_1 = x_2$. Therefore $f$ is injective.
Since $A = \mathbb{N} \setminus \{4, 5\}$ and $f(x) = x + 5$, we obtain that $\forall x \in A, f(x) \geq 6$. Hence choose $y = 1$ in the co-domain $A$, then we cannot find its preimage in the domain. Therefore, $f : A \to A$ is not surjective. Therefore $f$ is an injective but not surjective function, i.e., $f(A) \neq A$. By definition of infinite sets, we conclude that $A$ is an infinite set. □