Exercise 1. Prove or disprove the following statements involving universal quantifiers.

(a) \( \forall a \in U, \forall b \in U, \text{ if } (a \in \mathbb{Q}) \land (b \notin \mathbb{Q}), \text{ then } (a + b) \notin \mathbb{Q} \).

**Proof by contradiction:** For any two numbers \( a, b \) in \( U \), suppose that

\[
(a \in \mathbb{Q}) \land (a \notin \mathbb{Q}) \land (a + b) \in \mathbb{Q}
\]

\( a \in \mathbb{Q} \Rightarrow a = \frac{p}{q} \) where \( p, q \) are two integers

\( (a + b) \in \mathbb{Q} \Rightarrow a + b = \frac{s}{t} \) where \( s, t \) are two integers

\[\Rightarrow b = (a + b) - a = \frac{s}{t} - \frac{p}{q} = \frac{sq - pt}{tq}\]

Since \( s, t, p \) and \( q \) are integers, their product and difference are integers. Hence by definition of rational numbers, \( b \in \mathbb{Q} \). But as we suppose, \( b \notin \mathbb{Q} \). A contradiction. Therefore, for any two numbers \( a \) and \( b \), we prove our original statement.

(b) For any two nonempty sets \( A \) and \( B \), if \( A \setminus B = \emptyset \), then \( A \subseteq B \).

**Proof:** For any two nonempty sets \( A \) and \( B \), suppose \( A \setminus B = \emptyset \), then

\( A \setminus B = \emptyset \iff \neg(\exists x \text{ such that } x \in A \setminus B) \)

\[\iff \neg(\exists x \text{ such that } (x \in A) \land (x \notin B)) \]

\(\iff \forall x, (x \notin A) \lor (x \in B) \)

\(\iff \forall x, (x \in A) \rightarrow (x \in B) \)

Then, \( \forall x \in A, x \in B \), i.e., \( A \subseteq B \).

Since \( A, B \) are two arbitrary nonempty sets, it follows that if \( A \setminus B = \emptyset \), then \( A \subseteq B \).
(c) For any three natural numbers \( a, b \) and \( c \), if \( a \mid c \) and \( b \mid c \), then \((a \cdot b) \mid c\).

**Disproof:** Choose \( a = 2 \), \( b = 4 \) and \( c = 12 \), then it is true that \( 2 \mid 12 \) and \( 4 \mid 12 \), but it’s false that \( 8 \mid 12 \).

**Exercise #2.** Consider the following statement. For any three nonempty sets \( A, B \) and \( C \), if \( A \subseteq C \), and \( B \) and \( C \) are disjoint, then \( A \) and \( B \) are disjoint.

(a) Use Venn diagram to make sense of the statement.

(b) What do we need to assume to prove the statement? Express the assumptions without using \( \cap, \cup, \subseteq, \supseteq, \setminus, \) and \( = \).

We need to suppose all premises in the conditional statement, i.e., \( A, B \) and \( C \) are three nonempty sets.

\( A \subseteq C \iff \forall x \in A, x \in C \)

\( B \) and \( C \) are disjoint \( \iff \neg(\exists x \text{ such that } (x \in B) \land (x \in C)) \)

Apart from that, notice that the conclusion in the original conditional statement, which is \( A \) and \( B \) are disjoint, is equivalent to \( \neg(\exists x \text{ such that } (x \in A) \land (x \in B)) \iff \forall x, x \in A \rightarrow x \notin B. \)

Therefore, we need to prove a conditional statement which has another conditional statement in its conclusion, and we should also suppose the premise of that inner conditional statement, i.e., \( \forall x, x \in A. \)

(c) Prove the statement.

**Proof:** Suppose \( A, B \) and \( C \) are any three nonempty sets, \( A \subseteq C \), and \( B \) and \( C \) are disjoint.

Suppose \( \forall x \in U, x \in A, \) then \( ((x \in A) \land (A \subseteq C)) \Rightarrow x \in C \)

Also \((B \cap C = \emptyset) \iff \forall a \in C, a \notin B \)

\((x \in C) \land (B \cap C = \emptyset)) \Rightarrow x \notin B. \)

Therefore, given an arbitrary \( x, \) we prove that if \( x \in A \) then \( x \notin B. \)

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1In general, it’s said \( m \mid n \) (read as \( m \) divides \( n \)) for nonzero integers \( m \) and \( n \) if and only if there exists an integer \( k \) such that \( n = km \)
Exercise #3. Consider the following statement. For any three sets $A$, $B$ and $C$, if $A \subseteq B \setminus C$ and $A \neq \emptyset$, then $B$ is not a subset of $C$.

(a) Use Venn diagram to make sense of the statement.

(b) What do we need to assume to prove the statement? Express the assumptions without using $\cap$, $\cup$, $\subset$, $\subseteq$, $\supset$, $\supseteq$, \ and \ =. We need to assume:
   
   (a) $A$, $B$ and $C$ are three sets.
   (b) $A \subseteq B \setminus C$, which is equivalent to $\forall x \in A, x \in B \setminus C \iff \forall x \in A, (x \in B) \land (x \notin C)$
   (c) $A \neq \emptyset$, which is equivalent to $\exists x$ such that $x \in A$

(c) Rewrite “$B$ is not a subset of $C$” without using $\cap$, $\cup$, $\subset$, $\subseteq$, $\supset$, $\supseteq$, \ and \ =.

   $B$ is not a subset of $C$ is equivalent to $\neg (B \subseteq C) \iff \neg (\forall x \in B, x \in C) \iff \exists x \in B$ such that $x \notin C$.

(d) Prove the statement.

**Proof:** Suppose that $A$, $B$ and $C$ are three sets, $A \subseteq B \setminus C$ and $A \neq \emptyset$.

Note that $B$ is not a subset of $C \iff \neg (B \subseteq C) \iff \neg (\forall x \in U, x \notin B \lor x \notin C) \iff \exists x \in U$ s.t. $x \in B \land x \notin C$.

Since $A \neq \emptyset$, choose $a \in A$.

$a \in A \land A \subseteq (B \setminus C) \implies (a \in B \setminus C) \implies (a \in B \land a \notin C)$.

Therefore it follows that $\exists x \in U$ s.t. $x \in B \land x \notin C$.

Since $A$, $B$ and $C$ are any three sets, it follows that if $A \subseteq (B \setminus C)$ and $A \neq \emptyset$ then $B$ is not a subset of $C$. 

i.e., $A$ and $B$ are disjoint.