Gronwall’s Inequality

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1. Brief Introduction

Suppose $X$ is a Banach Space, and $f, g : [a, b] \times U \to X$ where $U \subset X$ is open. Then we can estimate the difference between two solutions respectively to $y' = f(t, y(t))$ and $z' = g(t, z(t))$ in terms of the difference between their initial conditions and that between $f$ and $g$. Usually Gronwall’s Inequality is usually used to obtain an estimate on $\|y(t) - z(t)\|$ when $f$ is not Lipschitz and not guaranteed to have a unique solution but $g$ is Lipschitz or satisfies certain smoothness conditions.

2. Gronwall’s Inequality

Theorem 2.1 (The Gronwall’s Inequality). Let $X$ be a Banach Space and $U \subset X$ an open set in $X$. Let $f, g : [a, b] \times U \to X$ be continuous functions and let $y, z : [a, b] \to X$ satisfy the initial value problems

(1) $y'(t) = f(t, y(t)), \quad y(0) = y_0,$

(2) $z'(t) = g(t, z(t)), \quad z(0) = z_0.$

Also assume that there is a constant $C$ such that

(3) $\|g(t, x_1) - g(t, x_2)\| \leq C \|x_1 - x_2\|$ for all $x_1, x_2 \in U$

and a continuous function $\varphi : [a, b] \to [0, \infty)$ so that

(4) $\|f(t, x) - g(t, x)\| \leq \varphi(t)$ for all $x \in U$

Then for $t \in [a, b]$

(5) $\|y(t) - z(t)\| \leq e^{C|t-a|} \|y_0 - z_0\| + e^{C|t-a|} \int_a^t e^{-C|s-a|} \varphi(s) ds.$
3. Proof of the Gronwall’s Inequality

Lemma 3.1. Let $X$ be a Banach Space, and $h(t) : [a, b] \to X$ be a $C^1$ function. Then we have

\[ \frac{d}{dt} \| f(t) \| \leq \| f'(t) \|. \tag{6} \]

Proof. Let $t \in [a, b)$ be arbitrary and $\delta$ be sufficiently small that $[t, t + \delta] \subset [a, b]$. Then by properties of norm, we obtain

\[ \| f(t + \delta) - f(t) \| \leq \| f(t + \delta) - f(t) \|. \]

It follows that

\[ \int_t^{t+\delta} \frac{d}{ds} \| f(s) \| \leq \int_t^{t+\delta} f'(s) \, ds. \]

Then let $\delta \to 0$, we thus obtain (6). \qed

3.1. proof of the inequality.

Proof. Using assumptions (2), (3), (4) we can see that

\[ \frac{d}{dt} \| y(t) - z(t) \| \leq \| y'(t) - z'(t) \| = \| f(t, y(t)) - g(t, z(t)) \| \leq \| f(t, y(t)) - g(t, y(t)) \| + \| g(t, y(t)) - g(t, z(t)) \| \leq \varphi(t) + C \| y(t) - z(t) \|. \]

That is

\[ \frac{d}{dt} \| y(t) - z(t) \| + C \| y(t) - z(t) \| \leq \varphi(t). \]

Multiplying both sides by $e^{-Ct}$ we obtain

\[ \frac{d}{dt} (e^{-Ct} \| y(t) - z(t) \|) \leq e^{-Ct} \varphi(t). \]

Integrating both sides from $a$ to $t$ we have

\[ e^{-Ct} \| y(t) - z(t) \| - e^{-Ca} \| y_0 - z_0 \| \leq \int_a^t e^{-Cs} \varphi(s) \, ds. \]

Hence we obtain the equivalence of (5). \qed