Theorem 0.1. Consider the second order ODE $y'' + q(x)y = 0$. If $q(x) \leq 0$ on an internal $I$, then non-trivial solutions of this ODE have at most one zero on $I$.

Proof. Suppose there exists a non-trivial solution $y$ that has at least two zeros on $I$. Without loss of generality let $x_1 < x_2$ be two consecutive zeros of $y$ on $I$, and also $y > 0$ on $(x_1, x_2)$. Then that tells us $y'(x_1) > 0$ and $y'(x_2) < 0$. This is indeed true. For otherwise, suppose $y'(x_1) \leq 0$. There are two cases.

(i) Case (1): If $y'(x_1) = 0$, then since $y'(x_1) = 0$, it follows that $y(x) \equiv 0$. This is a contradiction with $y(x) > 0$ for $x_1 < x < x_2$.

(ii) Case (2): If $y'(x_1) < 0$, then since $y'(x)$ is continuous on $[x_1, x_2]$, there exists $\delta > 0$ such that $\forall x \in [x_1, x_1 + \delta]$, $y'(x) < 0$. Therefore $y(x) = f'(x_0)(x - x_1) < 0$ where $x \in [x_1, x_1 + \delta]$ and $x_0 \in [x_1, x]$. This is a contradiction with $y(x) > 0$ for all $x_1 < x < x_2$.

By similar argument, $y'(x_2) < 0$.

However, since

$$y''(x) = -q(x)y(x) \geq 0, \forall x \in (x_1, x_2).$$

It follows that $y'(x)$ is non-decreasing, then $0 > y'(x_2) \geq y'(x_1) > 0$. This is a contradiction. Therefore the non-trivial solution of ODE $y'' + q(x)y = 0$ with $q(x) \leq 0$ on $I$ can have at most one zero. □

0.1. Strum’s Comparison Theorem.

Theorem 0.2 (Comparison). Let $\varphi_1$ and $\varphi_2$ be non-trivial solutions of equations

(1) $y'' + q_1(x)y = 0$

and

(2) $y'' + q_2(x)y = 0$

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respectively, on an interval $I$. And $q_1$ and $q_2$ are continuous functions on $I$ such that $q_1(x) \leq q_2(x)$. Then between any two consecutive zeros $x_1$ and $x_2$ of $\varphi_1$, there exists at least one zero of $\varphi_2$ unless $q_1(x) \equiv q_2(x)$ on $(x_1, x_2)$.

**Proof.** Suppose $x_1$ and $x_2$ are two consecutive zeros of $\varphi_1$ on $I$. Without loss of generality assume that $\varphi_1(x) > 0$ on $(x_1, x_2)$. Then by similar argument in Theorem 0.1 we know that $\varphi'_1(x_1) > 0$ and $\varphi'_2(x_2) < 0$.

Suppose to the contrary that $\varphi_2$ does not have a zero on $(x_1, x_2)$, then without loss of generality suppose $\varphi_2(x) > 0$ for all $x \in (x_1, x_2)$.

Then substituting $y$ in (1) for $\varphi_1$ and then multiplying (1) by $\varphi_2$, we have

$$\varphi''_1 \varphi_2 + q_1(x) \varphi_1 \varphi_2 = 0 \quad (3)$$

and

$$\varphi''_2 \varphi_1 + q_2(x) \varphi_2 \varphi_1 = 0 \quad (4)$$

Subtracting (3) and (4) gives us

$$(\varphi'_1 \varphi_2 - \varphi'_2 \varphi_1)' = (q_2(x) - q_1(x)) \varphi_1 \varphi_2.$$  

Integrating both sides from $x_1$ to $x_2$, we obtain

$$\varphi'_1(x_2) \varphi_2(x_2) - \varphi'_1(x_1) \varphi_2(x_1) = \int_{x_1}^{x_2} (q_2(x) - q_1(x)) \varphi_1 \varphi_2 dx.$$  

The LHS of the equation is always nonpositive, but the RHS is always positive unless $q_1(x) \equiv q_2(x)$ on $(x_1, x_2)$. Therefore, we have a contradiction if $q_1(x) \neq q_2(x)$ on $(x_1, x_2)$. 

\[\square\]

0.2. **Exercise.** Consider the ODE $y'' + y = 0$. It has $y = \sin(x)$ and $y = \cos(x)$ as two linearly independent solutions. Also both of them have infinitely many zeros on $\mathbb{R}$. Then equation $y'' + xy = 0$ have infinitely many zeros on $(1, \infty)$. Additionally, any of its non-trivial solutions have a zero between $n\pi$ and $(n+1)\pi$ for any natural number $n$. 