

PRESERVATION OF PRODUCT STRUCTURES UNDER THE RICCI FLOW WITH INSTANTANEOUS CURVATURE BOUNDS

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ABSTRACT. In this note, we prove that there exists a constant $\epsilon > 0$, depending only on the dimension, such that if a complete solution to the Ricci flow splits as a product at time $t = 0$ and has curvature bounded by $\frac{\epsilon}{t}$, then the solution splits for all time.

1. INTRODUCTION

In this note, we consider the problem of whether a solution to the Ricci flow

$$(1.1) \quad \frac{\partial}{\partial t} g = -2\text{Rc}$$

which splits as a product at $t = 0$ continues to do so for all time.

This problem is closely related, but not strictly equivalent, to the question of uniqueness of solutions to (1.1). For example, when $(\hat{M} \times \check{M}, \hat{g}_0 \oplus \check{g}_0)$, Shi's existence theorem [12] implies that there exist complete, bounded curvature solutions $(\hat{M}, \hat{g}(t))$ and $(\check{M}, \check{g}(t))$ with initial conditions \hat{g}_0 and \check{g}_0 , respectively, which exist on some common time interval $[0, T]$. Then, $\hat{g}(t) \oplus \check{g}(t)$ solves (1.1) on $\hat{M} \times \check{M}$ for $t \in [0, T]$ and is also complete and of bounded curvature. But, according to the uniqueness results of Hamilton [5] and Chen-Zhu [2], such a solution is unique among those which are complete and have bounded curvature. Thus, any solution in that class starting at $\hat{g}_0 \oplus \check{g}_0$ continues to split as a product.

Outside of this class, less is known. While there are elementary examples which show that without completeness, a solution may instantaneously cease to be a product, the extent to which the uniform curvature bound can be relaxed is less well-understood. (One exception is in dimension two, where the work of Giesen and Topping [3, 4] has established an essentially complete theory of existence and uniqueness for (1.1). In particular, in [14], Topping shows that any two complete solutions with the same initial data must agree.)

One class of particular interest is that of solutions satisfying a curvature bound of the form c/t for some constant c , which arise naturally as limits of exhaustions (see, e.g., [1], [6], [13]). The purpose of this note is to prove the following.

Theorem 1.1. *Let (\hat{M}, \hat{g}_0) and (\check{M}, \check{g}_0) be two Riemannian manifolds and let $M = \hat{M} \times \check{M}$ and $g_0 = \hat{g}_0 \oplus \check{g}_0$. Then there exists a constant $\epsilon = \epsilon(n) > 0$, where $n = \dim(M)$, such that if $g(t)$ is a complete solution to (1.1) on $M \times [0, T]$ with $g(0) = g_0$ satisfying*

$$(1.2) \quad |\text{Rm}| \leq \frac{\epsilon}{t},$$

then $g(t)$ splits as a product for all $t \in [0, T]$, i.e., $g(t) = \hat{g}(t) \oplus \check{g}(t)$, where $\hat{g}(t)$ and $\check{g}(t)$ are solutions to (1.1) on \hat{M} and \check{M} , respectively, for $t \in [0, T]$.

Lee [8] has already established the uniqueness of complete solutions satisfying the bound (1.2). However, his result does not directly imply Theorem 1.1: without any restrictions on the curvatures of \hat{g}_0 and \check{g}_0 , we lack the short-time existence theory to guarantee that there are *any* solutions on \hat{M} and \check{M} , respectively, with the given initial data, let alone solutions satisfying a bound of the form (1.2) for sufficiently small ϵ . Thus we are unable to construct a competing product solution on $\hat{M} \times \check{M}$ to which we might apply Lee's theorem.

Instead, we frame the problem as one of uniqueness for a related system, using a perspective similar to that of [9] and [11]. The key ingredient is a maximum principle closely based on one due to Huang-Tam [7] and modified by Liu-Székelyhidi [9]. These references establish, among other things, related results concerning the preservation of Kähler structures.

2. TRACKING THE PRODUCT STRUCTURE

Our first step toward proving Theorem 1 is to construct a system associated to a solution to Ricci flow which measures the degree to which a solution which initially splits as a product fails to remain a product. Consider a Riemannian product $(M, g_0) = (\hat{M} \times \check{M}, \hat{g}_0 \oplus \check{g}_0)$, and let $g(t)$ be a smooth solution to the Ricci flow on $M \times [0, T]$ with $g(0) = g_0$. For the time being, we make no assumptions on the completeness of $g(t)$ or bounds on its curvature.

2.1. Extending the projections. Let $\hat{\pi} : M \rightarrow \hat{M}$ and $\check{\pi} : M \rightarrow \check{M}$ be the projections on each factor, and let $\hat{H} = \ker(d\hat{\pi})$ and $\check{H} = \ker(d\check{\pi})$. We define $\hat{P}_0, \check{P}_0 \in \text{End}(TM)$ to be the orthogonal projections onto \hat{H} and \check{H} determined by g_0 .

Following [11], we extend each of them to a time-dependent family of projections for $t \in [0, T]$ by solving the fiber-wise ODEs

$$(2.1) \quad \begin{cases} \partial_t \hat{P}(t) = \text{Rc} \circ \hat{P} - \hat{P} \circ \text{Rc} \\ \hat{P}(0) = \hat{P}_0 \end{cases}, \quad \begin{cases} \partial_t \check{P}(t) = \text{Rc} \circ \check{P} - \check{P} \circ \text{Rc} \\ \check{P}(0) = \check{P}_0 \end{cases}.$$

From \hat{P} and \check{P} , we construct time-dependent endomorphisms $\mathcal{P}, \bar{\mathcal{P}} \in \text{End}(\Lambda^2 T^* M)$ by

$$\begin{aligned} \mathcal{P}\omega(X, Y) &= \omega(\hat{P}X, \check{P}Y) + \omega(\check{P}X, \hat{P}Y), \\ \bar{\mathcal{P}}\omega(X, Y) &= \omega(\hat{P}X, \hat{P}Y) + \omega(\check{P}X, \check{P}Y). \end{aligned}$$

Let $\text{Rm} : \Lambda^2 T^* M \rightarrow \Lambda^2 T^* M$ be the curvature operator, and define the following:

$$\begin{aligned} \mathcal{R} &= \text{Rm} \circ \mathcal{P}, & \bar{\mathcal{R}} &= \text{Rm} \circ \bar{\mathcal{P}}, \\ \mathcal{S} &= (\nabla \text{Rm}) \circ (\text{Id} \times \mathcal{P}), & \mathcal{T} &= (\nabla \nabla \text{Rm}) \circ (\text{Id} \times \text{Id} \times \mathcal{P}). \end{aligned}$$

In order to study the evolution of \mathcal{R} , it will be convenient to introduce an operator Λ_b^a which acts algebraically on tensors via

$$\Lambda_b^a A_{i_1 \dots i_l}^{j_1 \dots j_k} = \delta_{i_1}^a A_{b i_2 \dots i_l}^{j_1 \dots j_k} + \dots + \delta_{i_l}^a A_{i_1 \dots i_{l-1}}^{j_1 \dots j_k} - \delta_b^{j_1} A_{i_1 \dots i_l}^{a \dots j_k} - \dots - \delta_b^{j_k} A_{i_1 \dots i_l}^{j_1 \dots a}.$$

We will also consider the operator

$$D_t := \partial_t + R_{ab} g^{bc} \Lambda_c^a.$$

This operator has the property that $D_t g = 0$, and for any time-dependent tensor fields A and B ,

$$D_t \langle A, B \rangle = \langle D_t A, B \rangle + \langle A, D_t B \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the metric induced by $g(t)$. Note that by construction the projections satisfy

$$D_t \hat{\mathcal{P}} \equiv 0, \quad D_t \check{\mathcal{P}} \equiv 0, \quad D_t \mathcal{P} \equiv 0, \quad D_t \bar{\mathcal{P}} \equiv 0.$$

2.2. Evolution equations. In order to determine how the components of \mathbf{X} and \mathbf{Y} evolve, we will make use of the following commutation formulas (see [11], Lemma 4.3):

$$(2.2) \quad [D_t, \nabla_a] = \nabla_p R_{pacb} \Lambda_c^b + R_{ac} \nabla_c,$$

$$(2.3) \quad [D_t - \Delta, \nabla_a] = 2R_{abdc} \Lambda_d^c \nabla_b + 2R_{ab} \nabla_b.$$

Additionally, we will need to examine the sharp operator on endomorphisms of two forms. For any $A, B \in \text{End}(\Lambda^2 T^* M)$,

$$\langle A \# B(\varphi), \psi \rangle = \frac{1}{2} \sum_{\alpha, \beta} \langle [A(\omega_\alpha), B(\omega_\beta)], \varphi \rangle \cdot \langle [\omega_\alpha, \omega_\beta], \psi \rangle,$$

where $\varphi, \psi \in \Lambda^2 T^* M$ and $\{\omega_\alpha\}$ is an orthonormal basis for $\Lambda^2 T^* M$. Recall that the curvature operator evolves according to

$$(D_t - \Delta) \text{Rm} = \mathcal{Q}(\text{Rm}, \text{Rm}),$$

under the Ricci flow, where $\mathcal{Q}(A, B) = \frac{1}{2}(AB + BA) + A \# B$.

Proposition 2.1. *We have the following evolution equations for the projection \mathcal{P} :*

$$D_t \nabla \mathcal{P} = \text{Rm} * \nabla \mathcal{P} + \mathcal{P} * \mathcal{S},$$

$$D_t \nabla^2 \mathcal{P} = \text{Rm} * \nabla^2 \mathcal{P} + \nabla \text{Rm} * \nabla \mathcal{P} + \mathcal{P} * \mathcal{T} + \nabla \text{Rm} * \mathcal{P} * \nabla \mathcal{P}.$$

In particular, there exists a constant $C = C(n)$ such that

$$(2.4) \quad \begin{aligned} |D_t \nabla \mathcal{P}| &\leq C(|\text{Rm}| |\nabla \mathcal{P}| + |\mathcal{S}|), \\ |D_t \nabla^2 \mathcal{P}| &\leq C(|\text{Rm}| |\nabla^2 \mathcal{P}| + |\nabla \text{Rm}| |\nabla \mathcal{P}| + |\mathcal{T}|). \end{aligned}$$

Proof. Using equation (2.2) and the fact that $D_t \mathcal{P} = 0$, we can see that $D_t \nabla \mathcal{P} = [D_t, \nabla] \mathcal{P}$. With some additional computation, we can then see (as in Propositions 4.5 and 4.6 from [11]) that

$$D_t \nabla \mathcal{P} = \text{Rm} * \nabla \mathcal{P} + \mathcal{P} * \mathcal{S}.$$

Similarly, using this equation together with (2.2) and the fact that $\nabla \mathcal{S} = \mathcal{T} + \nabla \text{Rm} * \nabla \mathcal{P}$, we have

$$\begin{aligned} D_t \nabla^2 \mathcal{P} &= [D_t, \nabla] \nabla \mathcal{P} + \nabla (D_t \nabla \mathcal{P}) \\ &= \text{Rm} * \nabla^2 \mathcal{P} + \nabla \text{Rm} * \nabla \mathcal{P} + \mathcal{P} * \mathcal{T} + \nabla \text{Rm} * \mathcal{P} * \nabla \mathcal{P}, \end{aligned}$$

as claimed. \square

In order compute similar evolution equations for \mathcal{R}, \mathcal{S} , and \mathcal{T} , we will need the following lemma.

Lemma 2.2. *Let $A, B \in \text{End}(\Lambda^2 T^* M)$ be self-adjoint operators. There exists $C = C(n) > 0$ such that*

$$|\mathcal{Q}(A, B) \circ \mathcal{P}| \leq C(|A \circ \mathcal{P}| |B| + |A| |B \circ \mathcal{P}|).$$

Proof. Clearly,

$$|(A \circ B + B \circ A) \circ \mathcal{P}| \leq |A \circ \mathcal{P}| |B| + |A| |B \circ \mathcal{P}|.$$

Furthermore, for $\eta \in \Lambda^2 T^* M$,

$$\begin{aligned} ((A \# B) \circ \mathcal{P})(\eta) &= \frac{1}{2} \sum_{\alpha, \beta} \langle [A\omega_\alpha, B\omega_\beta], \mathcal{P}\eta \rangle \cdot [\omega_\alpha, \omega_\beta] \\ &= \frac{1}{2} \sum_{\alpha, \beta} \langle [\mathcal{P} \circ A\omega_\alpha, \mathcal{P} \circ B\omega_\beta], \mathcal{P}\eta \rangle \cdot [\omega_\alpha, \omega_\beta] \\ &\quad + \frac{1}{2} \sum_{\alpha, \beta} \langle [\bar{\mathcal{P}} \circ A\omega_\alpha, \mathcal{P} \circ B\omega_\beta], \mathcal{P}\eta \rangle \cdot [\omega_\alpha, \omega_\beta] \\ &\quad + \frac{1}{2} \sum_{\alpha, \beta} \langle [\mathcal{P} \circ A\omega_\alpha, \bar{\mathcal{P}} \circ B\omega_\beta], \mathcal{P}\eta \rangle \cdot [\omega_\alpha, \omega_\beta] \\ &\quad + \frac{1}{2} \sum_{\alpha, \beta} \langle [\bar{\mathcal{P}} \circ A\omega_\alpha, \bar{\mathcal{P}} \circ B\omega_\beta], \mathcal{P}\eta \rangle \cdot [\omega_\alpha, \omega_\beta], \end{aligned}$$

where $\{\omega_\alpha\}$ is an orthonormal basis for $\Lambda^2 T^* M$. The final term on the right hand side is zero (see [11], Lemma 3.5); the point is that the image of $\bar{\mathcal{P}}$ is closed under the bracket and is perpendicular to the image of \mathcal{P} . Moreover, $\mathcal{P} \circ A = (A \circ \mathcal{P})^*$ and $\mathcal{P} \circ B = (B \circ \mathcal{P})^*$, so it follows that

$$\begin{aligned} |(A \# B) \circ \mathcal{P}| &\leq C(|A \circ \mathcal{P}| |B \circ \mathcal{P}| + |A \circ \bar{\mathcal{P}}| |B \circ \mathcal{P}| + |A \circ \mathcal{P}| |B \circ \bar{\mathcal{P}}|) \\ &\leq C(|A \circ \mathcal{P}| |B| + |A| |B \circ \mathcal{P}|), \end{aligned}$$

completing the proof. \square

Proposition 2.3. *As defined above, \mathcal{R} , \mathcal{S} , and \mathcal{T} satisfy the inequalities*

$$\begin{aligned} (D_t - \Delta)\mathcal{R} &\leq C(|\text{Rm}||\mathcal{R}| + |\nabla \text{Rm}||\nabla \mathcal{P}| + |\text{Rm}||\nabla^2 \mathcal{P}|), \\ (2.5) \quad (D_t - \Delta)\mathcal{S} &\leq C(|\nabla \text{Rm}||\mathcal{R}| + |\text{Rm}||\mathcal{S}| + |\nabla^2 \text{Rm}||\nabla \mathcal{P}| + |\nabla \text{Rm}||\nabla^2 \mathcal{P}|), \\ (D_t - \Delta)\mathcal{T} &\leq C(|\nabla^2 \text{Rm}||\mathcal{R}| + |\nabla \text{Rm}||\mathcal{S}| + |\text{Rm}||\mathcal{T}| \\ &\quad + (|\nabla \text{Rm}||\text{Rm}| + |\nabla^3 \text{Rm}|)|\nabla \mathcal{P}| + |\nabla^2 \mathcal{R}||\nabla^2 \mathcal{P}|), \end{aligned}$$

where $C = C(n) > 0$.

Proof. Using the evolution equation for Rm , we have

$$(D_t - \Delta)\mathcal{R} = \mathcal{Q}(\text{Rm}, \text{Rm}) \circ \mathcal{P} + \text{Rm} \circ \Delta \mathcal{P} + 2\nabla \text{Rm} * \nabla \mathcal{P}.$$

The first inequality then follows immediately from Lemma 2.2.

We now compute the evolution equation for \mathcal{S} . First, note that

$$\begin{aligned} (2.6) \quad (D_t - \Delta)\mathcal{S} &= ([D_t - \Delta, \nabla] \text{Rm}) \circ \mathcal{P} + \nabla((D_t - \Delta)\text{Rm}) \circ \mathcal{P} \\ &\quad + \nabla^2 \text{Rm} * \nabla \mathcal{P} + \nabla \text{Rm} * \nabla^2 \mathcal{P}. \end{aligned}$$

For the first term, using the commutator (2.3), we have

$$[(D_t - \Delta), \nabla_a] R_{ijkl} = 2R_{abdc} \Lambda_d^c \nabla_b R_{ijkl} + 2R_{ab} \nabla_b R_{ijkl}.$$

As in the computation in Proposition 4.13 from [11], we have

$$R_{abdc} \Lambda_d^c \nabla_b R_{mnlk} \mathcal{P}_{ijmn} = \text{Rm} * \mathcal{S} + \nabla \text{Rm} * \mathcal{R} * \mathcal{P},$$

which gives us

$$(2.7) \quad ([D_t - \Delta, \nabla] \text{Rm}) \circ \mathcal{P} = \text{Rm} * \mathcal{S} + \nabla \text{Rm} * \mathcal{R} * \mathcal{P}.$$

We then compute

$$\begin{aligned} \nabla((D_t - \Delta) \text{Rm}) &= \nabla \mathcal{Q}(\text{Rm}, \text{Rm}) \\ &= \nabla \text{Rm} \circ \text{Rm} + \text{Rm} \circ \nabla \text{Rm} + \nabla \text{Rm} \# \text{Rm} + \text{Rm} \# \nabla \text{Rm} \\ &= 2\mathcal{Q}(\nabla \text{Rm}, \text{Rm}), \end{aligned}$$

where we regard ∇Rm as a one form with values in $\text{Sym}(\Lambda^2 T^* M)$. Then, applying Lemma 2.2 and combining the result in (2.7) with (2.6), we obtain the second inequality.

For the third inequality, we begin with the identity

$$(D_t - \Delta) \mathcal{T} = ((D_t - \Delta) \nabla^2 \text{Rm}) \circ \mathcal{P} + \nabla^2 \text{Rm} * \nabla^2 \mathcal{P} + \nabla^3 \text{Rm} * \nabla \mathcal{P}.$$

The first term can be rewritten as

$$\begin{aligned} ((D_t - \Delta) \nabla^2 \text{Rm}) \circ \mathcal{P} &= ([D_t - \Delta, \nabla] \nabla \text{Rm}) \circ \mathcal{P} + (\nabla [D_t - \Delta, \nabla] \text{Rm}) \circ \mathcal{P} \\ &\quad + (\nabla \nabla (D_t - \Delta) \text{Rm}) \circ \mathcal{P}. \end{aligned}$$

Applying equation (2.3) once again gives us

$$((D_t - \Delta) \nabla_a \nabla \text{Rm}) \circ \mathcal{P} - (\nabla_a (D_t - \Delta) \nabla \text{Rm}) \circ \mathcal{P} = (2R_{abdc} \Lambda_d^c \nabla_b \nabla \text{Rm} + 2R_{ab} \nabla_b \nabla \text{Rm}) \circ \mathcal{P},$$

and we have

$$R_{abdc} \Lambda_d^c \nabla_b \nabla \text{Rm} \circ \mathcal{P} = \text{Rm} * \mathcal{T} + \nabla^2 \text{Rm} * \mathcal{R} * \mathcal{P}$$

(again see [11], Proposition 4.13, also [9]). We can see that

$$\begin{aligned} (\nabla [D_t - \Delta, \nabla] \text{Rm}) \circ \mathcal{P} &= \nabla(([D_t - \Delta, \nabla] \text{Rm}) \circ \mathcal{P}) + ([D_t - \Delta, \nabla] \text{Rm}) * \nabla \mathcal{P} \\ &= \nabla \text{Rm} * \mathcal{S} + \text{Rm} * \mathcal{T} + \text{Rm} * \nabla \text{Rm} * \nabla \mathcal{P} + \nabla^2 \text{Rm} * \mathcal{R} * \mathcal{P} + \nabla \text{Rm} * \mathcal{S} * \mathcal{P} \\ &\quad + \nabla \text{Rm} * \text{Rm} * \nabla \mathcal{P} * \mathcal{P} + \nabla \text{Rm} * \mathcal{R} * \nabla \mathcal{P} + \text{Rm} * \nabla \text{Rm} * \nabla \mathcal{P} \end{aligned}$$

where we again use the facts that $\nabla \mathcal{R} = \mathcal{S} + \text{Rm} * \nabla \mathcal{P}$ and $\nabla \mathcal{S} = \mathcal{T} + \nabla \text{Rm} * \nabla \mathcal{P}$. Additionally,

$$(\nabla \nabla (D_t - \Delta) \text{Rm}) \circ \mathcal{P} = 2\mathcal{Q}(\nabla^2 \text{Rm}, \text{Rm}) \circ \mathcal{P} + 2\mathcal{Q}(\nabla \text{Rm}, \nabla \text{Rm}) \circ \mathcal{P}.$$

Combining the above identities and again applying Lemma 2.2 to the last term, we obtain the third inequality. \square

2.3. Constructing a PDE-ODE system. With an eye toward Theorem 1, we now organize the tensors $\nabla \mathcal{P}$, $\nabla^2 \mathcal{P}$, \mathcal{R} , \mathcal{S} , and \mathcal{T} into groupings which satisfies a closed system of differential inequalities. Let

$$\mathcal{X} = \mathcal{T}^4(T^* M) \oplus \mathcal{T}^5(T^* M) \oplus \mathcal{T}^6(T^* M), \quad \mathcal{Y} = \mathcal{T}^5(T^* M) \oplus \mathcal{T}^6(T^* M),$$

and define families of sections $\mathbf{X} = \mathbf{X}(t)$ of \mathcal{X} and $\mathbf{Y} = \mathbf{Y}(t)$ of \mathcal{Y} for $t \in (0, T]$ by

$$(2.8) \quad \mathbf{X} = \left(\frac{\mathcal{R}}{t}, \frac{\mathcal{S}}{t^{1/2}}, \mathcal{T} \right), \quad \mathbf{Y} = \left(\frac{\nabla \mathcal{P}}{\sqrt{t}}, \nabla^2 \mathcal{P} \right).$$

Proposition 2.4. *If $g(t)$ is a smooth solution to Ricci flow on $M \times [0, T]$ with $|\text{Rm}|(x, t) < a/t$ for some $a > 0$, then there exists a constant $C = C(a, n) > 0$ depending such that \mathbf{X} and \mathbf{Y} satisfy*

$$|(D_t - \Delta) \mathbf{X}| \leq C \left(\frac{1}{t} |\mathbf{X}| + \frac{1}{t^2} |\mathbf{Y}| \right), \quad |D_t \mathbf{Y}| \leq C \left(|\mathbf{X}| + \frac{1}{t} |\mathbf{Y}| \right),$$

on $M \times (0, T]$.

Remark 2.5. *Inspection of the proof reveals that the constant C in fact has the form $C = a\tilde{C}$, where \tilde{C} depends only on n and $\max\{a, 1\}$.*

This follows directly from Propositions 2.1 and 2.3 with the help of the following curvature bounds, which can be obtained from the classical estimates of Shi [12] with a simple rescaling argument.

Proposition 2.6. *Suppose $(M, g(t))$ is a complete solution to Ricci flow for $t \in [0, T]$ which satisfies*

$$|\text{Rm}|(x, t) \leq \frac{a}{t}$$

for some constant $a > 0$. Then for each $m > 0$, there exists a constant $C = C(m, n)$ such that

$$|\nabla^{(m)} \text{Rm}|(x, t) \leq \frac{aC}{t^{m/2+1}}(1 + a^{m/2}).$$

Proof of Proposition 2.4. Throughout this proof, C will denote a constant which may change from line to line but depends only on n and a . Using (2.4) in combination with the curvature estimates, we obtain

$$\begin{aligned} |D_t \mathbf{Y}| &\leq \frac{1}{2}t^{-3/2}|\nabla \mathcal{P}| + t^{-1/2}|D_t \nabla \mathcal{P}| + |D_t \nabla^2 \mathcal{P}| \\ &\leq Ct^{-1/2}|\mathcal{S}| + C|\mathcal{T}| + Ct^{-3/2}|\nabla \mathcal{P}| + Ct^{-1}|\nabla^2 \mathcal{P}| \\ &\leq C|\mathbf{X}| + \frac{C}{t}|\mathbf{Y}|. \end{aligned}$$

Applying the curvature estimates to the inequalities (2.5) for \mathcal{R} , \mathcal{S} , and \mathcal{T} , we get

$$\begin{aligned} |(D_t - \Delta)\mathcal{R}| &\leq Ct^{-1}|\mathcal{R}| + Ct^{-3/2}|\nabla \mathcal{P}| + Ct^{-1}|\nabla^2 \mathcal{P}|, \\ |(D_t - \Delta)\mathcal{S}| &\leq Ct^{-3/2}|\mathcal{R}| + Ct^{-1}|\mathcal{S}| + Ct^{-2}|\nabla \mathcal{P}| + Ct^{-3/2}|\nabla^2 \mathcal{P}|, \end{aligned}$$

and

$$|(D_t - \Delta)\mathcal{T}| \leq Ct^{-2}|\mathcal{R}| + Ct^{-3/2}|\mathcal{S}| + Ct^{-1}|\mathcal{T}| + Ct^{-5/2}|\nabla \mathcal{P}| + t^{-2}|\nabla^2 \mathcal{P}|.$$

Combining these equations, we have

$$\begin{aligned} |(D_t - \Delta)\mathbf{X}| &\leq t^{-1}|(D_t - \Delta)\mathcal{R}| + t^{-2}|\mathcal{R}| + t^{-1/2}|(D_t - \Delta)\mathcal{S}| + \frac{1}{2}t^{-3/2}|\mathcal{S}| \\ &\quad + |(D_t - \Delta)\mathcal{T}| \\ &\leq Ct^{-2}|\mathcal{R}| + Ct^{-3/2}|\mathcal{S}| + Ct^{-1}|\mathcal{T}| + Ct^{-5/2}|\nabla \mathcal{P}| + Ct^{-2}|\nabla^2 \mathcal{P}| \\ &\leq Ct^{-1}|\mathbf{X}| + Ct^{-2}|\mathbf{Y}|, \end{aligned}$$

as desired. \square

3. A GENERAL UNIQUENESS THEOREM FOR PDE-ODE SYSTEMS

We now aim to show that \mathbf{X} and \mathbf{Y} vanish using a maximum principle from [7] by adapting it to apply to a general PDE-ODE system. The following theorem is essentially a reformulation of Lemma 2.3 in [7] and Lemma 2.1 in [9].

Theorem 3.1. *Let $M = M^n$ and \mathcal{X} and \mathcal{Y} be finite direct sums of $T_l^k(M)$. There exists an $\epsilon = \epsilon(n) > 0$ with the following property: Whenever $g(t)$ is a smooth, complete solution to the Ricci flow on M satisfying*

$$|\text{Rm}| \leq \frac{\epsilon}{t}$$

on $M \times (0, T]$, and $\mathbf{X} = \mathbf{X}(t)$ and $\mathbf{Y} = \mathbf{Y}(t)$ are families of smooth sections of \mathcal{X} and \mathcal{Y} satisfying

$$|(D_t - \Delta)\mathbf{X}| \leq \frac{C}{t}|\mathbf{X}| + \frac{C}{t^2}|\mathbf{Y}|, \quad |D_t\mathbf{Y}| \leq C|\mathbf{X}| + \frac{C}{t}|\mathbf{Y}|,$$

$$D_t^k\mathbf{Y} = 0, \quad D_t^k\mathbf{X} = 0 \text{ for } k \geq 0 \text{ at } t = 0,$$

and

$$|\mathbf{X}| \leq Ct^{-l},$$

for some $C > 0$, $l > 0$, then $\mathbf{X} \equiv 0$ and $\mathbf{Y} \equiv 0$ on $M \times [0, T]$.

The key ingredient in the proof of Theorem 3.1 is an the following scalar maximum principle due to Huang-Tam [7] (and its variant in [9]). Though the statement has been slightly changed from its appearance in [7], the proof is nearly identical. We detail here the modifications we make for completeness.

Proposition 3.2 (c.f. [7], Lemma 2.3 and [9], Lemma 2.1). *Let M be a smooth n -dimensional manifold. There exists an $\epsilon = \epsilon(n) > 0$ such that the following holds: Whenever $g(t)$ is a smooth complete solution to the Ricci flow on $M \times [0, T]$ such that the curvature satisfies $|\text{Rm}| \leq \epsilon/t$ for some and $f \geq 0$ is a smooth function on $M \times [0, T]$ satisfying*

$$(1) (\partial_t - \Delta)f(x, t) \leq at^{-1} \max_{0 \leq s \leq t} f(x, s),$$

$$(2) \partial_t^k \Big|_{t=0} f = 0 \text{ for all } k \geq 0,$$

$$(3) \sup_{x \in M} f(x, t) \leq Ct^{-l} \text{ for some positive integer } l \text{ for some constant } C,$$

then $f \equiv 0$ on $M \times [0, T]$.

Proof. For the time-being, we will assume $\epsilon > 0$ is fixed and that $g(t)$ is a smooth, complete solution to Ricci flow on $M \times [0, T]$ satisfying $|\text{Rm}| \leq \epsilon/t$. We will then specify ϵ over the course of the proof.

As in [7] we may assume $T \leq 1$. We will first show that for any $k > a$, there exists a constant B_k such that

$$\sup_{x \in M} f(x, t) \leq B_k t^k.$$

Let ϕ be a cutoff function as in [7], i.e., choose $\phi \in C^\infty([0, \infty))$ such that $0 \leq \phi \leq 1$ and

$$\phi(s) = \begin{cases} 1 & 0 \leq s \leq 1, \\ 0 & 2 \leq s, \end{cases} \quad -C_0 \leq \phi' \leq 0, \quad |\phi''| \leq C_0,$$

for some constant $C_0 > 0$. Then let $\Phi = \phi^m$ for $m > 2$ to be chosen later and define $q = 1 - \frac{2}{m}$. Then

$$0 \geq \Phi' \geq -C(m)\Phi^q, \quad |\Phi''| \leq C(m)\Phi^q.$$

where $C(m) > 0$ is a constant depending only on m (and on C_0).

Fix a point $y \in M$. As in Lemma 2.2 of [7], there exists some $\rho \in C^\infty(M)$ such that

$$d_{g(T)}(x, y) + 1 \leq \rho(x) \leq C'(d_{g(T)}(x, y) + 1), \quad |\nabla_{g(T)}\rho|_{g(T)} + |\nabla_{g(T)}^2\rho|_{g(T)} \leq C',$$

where C' is a positive constant depending only on n and $\frac{\epsilon}{T}$. This function then also satisfies

$$|\nabla\rho| \leq C_1 t^{-c\epsilon}, \quad |\Delta\rho| \leq C_2 t^{-1/2-c\epsilon},$$

where C_1, C_2 are constants depending only on n, T and ϵ , and $c > 0$ depends only on the dimension n . We may assume ϵ is small enough so that $c\epsilon < 1/4$. Let $\Psi(x) = \Psi_r(x) = \Phi(\rho(x)/r)$ for $r \gg 1$. Define also $\theta = \exp(-\alpha t^{1-\beta})$, where $\alpha > 0$ and $0 < \beta < 1$. By the estimates on the derivatives of ρ , we have

$$|\nabla\Psi| = r^{-1}|\Phi'(\rho/r)||\nabla\rho| \leq r^{-1}C(m)C_1\Phi^q(\rho/r)t^{-c\epsilon} \leq C(m)\Psi^q t^{-1/4}$$

and

$$\begin{aligned} |\Delta\Psi| &= |r^{-2}\Phi''(\rho/r)|\nabla\rho|^2 + r^{-1}\Phi'(\rho/r)\Delta\rho| \\ &\leq r^{-2}C(m)\Phi^q(\rho/r)t^{-2c\epsilon} + r^{-1}C(m)\Phi^q(\rho)t^{-1/2-c\epsilon} \\ &\leq C(m)\Psi^q t^{-3/4}. \end{aligned}$$

For $k > a$, let $F = t^{-k}f$. Then F satisfies

$$\begin{aligned} (\partial_t - \Delta)F &= -kt^{-k-1}f + t^{-k}(\partial_t - \Delta)f \\ &\leq -kt^{-k-1}f(x, t) + at^{-k-1} \max_{0 \leq s \leq t} f(x, s) \end{aligned}$$

and $F \leq Ct^{-l-k}$.

Let $H = \theta\Psi F$ and suppose that H attains a positive maximum at the point (x_0, t_0) . Then, at this point, we have $\Psi > 0$ and both $(\partial_t - \Delta)H \geq 0$ and $\nabla H = 0$. Since $\nabla H = 0$, we have

$$\nabla\Psi \cdot \nabla F = -\frac{F|\nabla\Psi|^2}{\Psi}.$$

Additionally, since Ψ is independent of time,

$$\theta(s)F(x_0, s) \leq \theta(t_0)F(x_0, t_0)$$

for all $s \leq t_0$. Because θ is decreasing, we have

$$s^{-k}f(x_0, s) = F(x_0, s) \leq F(x_0, t_0) = t_0^{-k}f(x_0, t_0)$$

for $s \leq t_0$. which in turn implies

$$\max_{0 \leq s \leq t_0} f(x, s) = f(x, t_0).$$

Thus, at (x_0, t_0) we have

$$(\partial_t - \Delta)F \leq (-k + a)t_0^{-1}F \leq 0.$$

Thus at (x_0, t_0) we have

$$\begin{aligned} \Delta H &= \theta F \Delta\Psi + \theta\Psi \Delta F + 2\theta \nabla F \cdot \nabla\Psi \\ &= \theta F \Delta\Psi + \theta\Psi \Delta F - 2\theta \frac{F|\nabla\Psi|^2}{\Psi} \\ &\geq -C(m)\theta F \Psi^q t_0^{-3/4} - C(m)\theta F \Psi^{2q-1} t_0^{-3/4} + \theta\Psi \Delta F \end{aligned}$$

and

$$\partial_t H = -\alpha(1-\beta)t_0^{-\beta}\theta\Psi F + \theta\Psi \partial_t F.$$

We can then compute

$$\begin{aligned}
0 &\leq (\partial_t - \Delta)H \\
&\leq \theta \Psi (\partial_t - \Delta)F - \alpha(1 - \beta)t_0^{-\beta} \theta \Psi F + C(m)\theta \Psi^q F t_0^{-3/4} + C(m)\theta \Psi^{2q-1} F t_0^{-3/4} \\
&\leq -\alpha(1 - \beta)t_0^{-\beta} \theta \Psi F + C(m)\theta (\Psi F)^q t_0^{-3/4 - (1-q)(l+k)} \\
&\quad + C(m)\theta (\Psi F)^{2q-1} t_0^{-3/4 - (2-2q)(l+k)}.
\end{aligned}$$

We now choose m and β so that the powers of t_0 in the denominators of the last two terms are less than β . We take β to be $7/8$ (any $\beta \in (3/4, 1)$ will do). Recalling that $q = 1 - 2/m$, we choose m large enough so that $7/8 > 3/4 + (1 - q)(l + k)$ and $7/8 > 3/4 + (2 - 2q)(l + k)$. Then

$$\frac{\alpha}{8} \Psi F = \alpha(1 - \beta) \Psi F \leq C(m) ((\Psi F)^q + (\Psi F)^{2q-1}).$$

Finally, we choose α large enough so that $\alpha > 16C(m)$. Then

$$2\Psi F \leq (\Psi F)^q + (\Psi F)^{2q-2},$$

implying that $(\Psi F)(x_0, t_0) \leq 1$, and hence $H \leq 1$ everywhere. In particular, for any $x \in \{\rho \leq r\}$, $f(x, t) = t^k F(x, t) \leq e^{\alpha t^k} := B_k t^k$. Sending r to infinity then proves that $f(x, t) \leq B_k t^k$.

Next, again as in [7], we define the function $\eta(x, t) = \rho(x) \exp\left(\frac{2C_2}{1-b} t^{1-b}\right)$ for $b > 1$. Since $|\Delta\rho| \leq C_2 t^{-b}$, we have

$$(\partial_t - \Delta)\eta > 0, \quad \partial_t \eta > 0.$$

Let $F = t^{-a} f$. Fix $\delta > 0$ and consider the function $F - \delta\eta - \delta t$. Note that by our previous argument, $F \leq Ct^2$, and in particular is bounded. For some $t_1 > 0$ depending on δ and c , $F - \delta t < 0$ for $t \leq t_1$ and for $t \geq t_1$, $F - \delta\eta < 0$ outside some compact set. So, if $F - \delta\eta - \delta t$ is ever positive, there must exist some $(x_0, t_0) \in M \times (0, T]$ at which it attains a positive maximum. Because $-\delta\eta - \delta t$ is decreasing in time, for any $s < t_0$ from the inequality

$$F(x_0, s) - \delta\eta(x_0, s) - \delta s \leq F(x_0, t_0) - \delta\eta(x_0, t_0) - \delta t_0,$$

we conclude

$$F(x_0, s) \leq F(x_0, t_0).$$

As in our previous argument, this implies that $f(x_0, t_0) = \max_{0 \leq s \leq t_0} f(x_0, s)$, so that at (x_0, t_0)

$$(\partial_t - \Delta)(F - \delta\eta - \delta t) < 0,$$

a contradiction. Thus, for any $\delta > 0$, $F - \delta\eta - \delta t \leq 0$. Taking $\delta \rightarrow 0$ then implies that $F = 0$. \square

We can now prove Theorem 3.1.

Proof of Theorem 3.1. For $k > 0$ to be determined later, define the functions F and G on $M \times [0, T]$ by $F = t^{-k} |\mathbf{X}|^2$, $G = t^{-(k+1)} |\mathbf{Y}|^2$ for $t \in (0, T]$ and $F(x, 0) = G(x, 0) = 0$. From the assumption that $D_t^l \mathbf{X} = D_t^l \mathbf{Y} = 0$ for all $l \geq 0$, it follows that both F and G are smooth on $M \times [0, T]$ and that $\partial_t^l F = \partial_t^l G = 0$ for all $l \geq 0$.

We have

$$\begin{aligned}
(\partial_t - \Delta)F &= -kt^{-(k+1)}|\mathbf{X}|^2 + 2t^{-k}\langle(D_t - \Delta)\mathbf{X}, \mathbf{X}\rangle - 2t^{-k}|\nabla\mathbf{X}|^2 \\
&\leq -kt^{-(k+1)}|\mathbf{X}|^2 + 2t^{-k}|(D_t - \Delta)\mathbf{X}||\mathbf{X}| \\
&\leq t^{-(k+1)}(2C - k)|\mathbf{X}|^2 + 2Ct^{-(k+2)}|\mathbf{X}||\mathbf{Y}| \\
&\leq t^{-1}(3C - k)F + Ct^{-2}G
\end{aligned}$$

and

$$\begin{aligned}
\partial_t G &= -(k+1)t^{-(k+2)}|\mathbf{Y}|^2 + 2t^{-(k+1)}\langle D_t \mathbf{Y}, \mathbf{Y} \rangle \\
&\leq (2C - k - 1)t^{-(k+2)}|\mathbf{Y}|^2 + 2Ct^{-(k+1)}|\mathbf{X}||\mathbf{Y}| \\
&\leq CF + t^{-1}(3C - k - 1)G.
\end{aligned}$$

Choosing $k > 3C$, this becomes

$$(\partial_t - \Delta)F \leq t^{-2}CG, \quad \partial_t G \leq CF.$$

In particular this implies that

$$G(x, t) \leq Ct \max_{0 \leq s \leq t} F(x, s),$$

and therefore

$$(\partial_t - \Delta)F \leq t^{-1}C^2 \max_{0 \leq s \leq t} F(x, s).$$

By our assumption on \mathbf{X} , $F \leq Ct^{-2l-k}$. Thus F satisfies the hypotheses of Proposition 3.2, and must vanish identically. We then conclude that G , hence \mathbf{Y} , vanishes as well. \square

4. PROOF OF THEOREM 1.1

We are now almost ready to prove Theorem 1.1. We just need to first verify that \mathbf{X} and \mathbf{Y} satisfy the last major remaining hypothesis of Theorem 3.1, that is, that all time derivatives of \mathbf{X} and \mathbf{Y} vanish at $t = 0$.

4.1. Vanishing of time derivatives. We begin by recording a standard commutator formula, which is in fact valid (with obvious modifications) for any family of smooth metrics.

Proposition 4.1. *Let $(M, g(t))$ be a smooth solution to the Ricci flow for $t \in [0, T]$. Then, for any $l \geq 1$, the formula*

$$(4.1) \quad [D_t, \nabla^{(l)}]\mathcal{A} = \sum_{k=1}^l \nabla^{(k-1)}[D_t, \nabla]\nabla^{(l-k)}\mathcal{A}$$

is valid for any smooth family of tensor fields \mathcal{A} on $M \times [0, T]$.

Proof. We proceed by induction on l . The base case, $l = 1$, is trivial. Now, suppose that (4.1) holds for $l \leq m$ for some $m \geq 1$. Then,

$$\begin{aligned}
[D_t, \nabla^{(m+1)}]\mathcal{A} &= D_t \nabla^{(m+1)}\mathcal{A} - \nabla^{(m+1)}D_t\mathcal{A} \\
&= [D_t, \nabla^{(m)}]\nabla\mathcal{A} + \nabla^m D_t \nabla\mathcal{A} - \nabla^{(m+1)}D_t\mathcal{A} \\
&= [D_t, \nabla^{(m)}]\nabla\mathcal{A} + \nabla^{(m)}[D_t, \nabla]\mathcal{A} \\
&= \sum_{k=1}^m \nabla^{(k-1)}[D_t, \nabla]\nabla^{(m-k)}(\nabla\mathcal{A}) + \nabla^{(m)}[D_t, \nabla]\mathcal{A} \\
&= \sum_{k=1}^{m+1} \nabla^{(k-1)}[D_t, \nabla]\nabla^{(m+1-k)}\mathcal{A},
\end{aligned}$$

as desired. \square

Now we argue inductively that $D_t^k \mathbf{X} = 0$ and $D_t^k \mathbf{Y} = 0$ at $t = 0$.

Proposition 4.2. *Let $M = \hat{M} \times \check{M}$ be a smooth manifold and $g(t)$ be a smooth, complete solution to the Ricci flow such that $g(0)$ splits as a product. Define \mathcal{P} and \mathcal{R} as in Section 2. The following equations hold at $t = 0$ for all $k, l \geq 0$:*

$$D_t^k \nabla^{(l)}\mathcal{R} = 0, \quad D_t^k \nabla^{(l+1)}\mathcal{P} = 0.$$

Proof. We proceed by induction on k , beginning with the base case $k = 0$. Because the metric splits as a product initially, at $t = 0$ we have $\nabla^{(l)}\hat{P} \equiv \nabla^{(l)}\check{P} \equiv 0$ for all $l \geq 0$ and

$$R(\hat{P}(\cdot), \check{P}(\cdot), \cdot, \cdot) \equiv 0.$$

From this we get that, for any $X, Y, Z, W \in TM$,

$$\mathcal{R}(X^* \wedge Y^*)(Z, W) = 2R(\hat{P}X, \check{P}Y, W, Z) + 2R(\check{P}X, \hat{P}Y, W, Z) = 0.$$

Combining these facts, we conclude

$$\nabla^{(l+1)}\mathcal{P} \equiv 0, \quad \nabla^{(l)}\mathcal{R} \equiv 0, \quad \nabla^{(l)}\mathcal{R}^* \equiv 0,$$

at $t = 0$, where $\mathcal{R}^* = \mathcal{P} \circ \text{Rm}$ denotes the adjoint of \mathcal{R} with respect to g .

Now starting the induction step, suppose that for some $k \geq 0$, for all $l \geq 0$ and any $m \leq k$,

$$D_t^m \nabla^{(l+1)}\mathcal{P} = 0, \quad D_t^m \nabla^{(l)}\mathcal{R} = 0,$$

hence also $D_t^m \nabla^{(l)}\mathcal{R}^* = 0$. Recall that

$$(D_t - \Delta)\text{Rm} = \mathcal{Q}(\text{Rm}, \text{Rm}).$$

As in [11], Lemma 4.9, $\mathcal{Q}(\text{Rm}, \text{Rm}) \circ \mathcal{P} = \mathcal{R} * \mathcal{U}_1 + \mathcal{R}^* * \mathcal{U}_2$, where \mathcal{U}_1 and \mathcal{U}_2 are smooth families of tensors on M . Thus we can compute

$$\begin{aligned}
D_t \mathcal{R} &= (D_t \text{Rm}) \circ \mathcal{P} + \text{Rm} \circ (D_t \mathcal{P}) \\
&= (\Delta \text{Rm}) \circ \mathcal{P} + \mathcal{Q}(\text{Rm}, \text{Rm}) \circ \mathcal{P},
\end{aligned}$$

and thus

$$(4.2) \quad D_t^{k+1} \mathcal{R} = D_t^k ((\Delta \text{Rm}) \circ \mathcal{P}) + D_t^k (\mathcal{Q}(\text{Rm}, \text{Rm}) \circ \mathcal{P}).$$

Because

$$\Delta \mathcal{R} = (\Delta \text{Rm}) \circ \mathcal{P} + \text{Rm} \circ \Delta \mathcal{P} + 2\nabla_i \text{Rm} \circ \nabla_i \mathcal{P},$$

by the induction hypothesis $D_t^k((\Delta\text{Rm}) \circ \mathcal{P}) \equiv 0$ at $t = 0$. Similarly,

$$D_t^k(\mathcal{Q}(\text{Rm}, \text{Rm}) \circ \mathcal{P}) = D_t^k(\mathcal{R} * \mathcal{U}_1) + D_t^k(\mathcal{R}^* * \mathcal{U}_2) = 0.$$

We conclude that $D_t^{k+1}\mathcal{R} \equiv 0$, and thus $D_t^{k+1}\mathcal{R}^* \equiv 0$.

Now, using the commutator from equation (2.2) and Proposition 4.1, for any $l > 0$ we have

$$\begin{aligned} D_t \nabla^{(l)} \mathcal{R} &= \sum_{m=1}^l \nabla^{(m-1)} [D_t, \nabla] \nabla^{(l-m)} \mathcal{R} + \nabla^{(l)} D_t \mathcal{R} \\ &= \sum_{m=1}^l \nabla^{(m-1)} \left(\nabla \text{Rm} * \nabla^{(l-m)} \mathcal{R} + \text{Rm} * \nabla^{(l-m+1)} \mathcal{R} \right) + \nabla^{(l)} D_t \mathcal{R}, \end{aligned}$$

and thus

$$D_t^{k+1} \nabla^{(l)} \mathcal{R} = \sum_{m=1}^l D_t^k \nabla^{(m-1)} \left(\nabla \text{Rm} * \nabla^{(l-m)} \mathcal{R} + \text{Rm} * \nabla^{(l-m+1)} \mathcal{R} \right) + D_t^k \nabla^{(l)} D_t \mathcal{R}.$$

Expanding using the product rule and applying the induction hypothesis, all terms in the first sum vanish at $t = 0$. For the remaining term, we again use the evolution equation for \mathcal{R} . We have

$$D_t^k \nabla^{(l)} D_t \mathcal{R} = D_t^k \nabla^{(l)} ((\Delta\text{Rm}) \circ \mathcal{P} + \mathcal{Q}(\text{Rm}, \text{Rm}) \circ \mathcal{P}).$$

As before, rewriting $\mathcal{Q}(\text{Rm}, \text{Rm}) \circ \mathcal{P}$ in terms of \mathcal{R} and \mathcal{R}^* and expanding using the product rule, it follows that $D_t^k \nabla^{(l)} D_t \mathcal{R} \equiv 0$ at $t = 0$.

We now move on to the derivatives of \mathcal{P} . Recall that

$$D_t \nabla \mathcal{P} = [D_t, \nabla] \mathcal{P} = \text{Rm} * \nabla \mathcal{P} + \mathcal{P} * \mathcal{S}.$$

Applying this in combination with Proposition 4.1, we get, for any $l \geq 1$,

$$\begin{aligned} D_t^{k+1} \nabla^{(l)} \mathcal{P} &= \sum_{m=1}^l D_t^k \nabla^{(m-1)} [D_t, \nabla] \nabla^{(l-m)} \mathcal{P} + D_t^k \nabla^{(l)} D_t \mathcal{P} \\ &= \sum_{m=1}^{l-1} D_t^k \nabla^{(m-1)} (\nabla \text{Rm} * \nabla^{(l-m)} \mathcal{P} + \text{Rm} * \nabla^{(l-m+1)} \mathcal{P}) \\ &\quad + D_t^k \nabla^{(l-1)} [D_t, \nabla] \mathcal{P} + D_t^k \nabla^{(l)} D_t \mathcal{P}. \end{aligned}$$

As before, every term in the first sum vanishes by the induction hypothesis, while the final term vanishes because $D_t \mathcal{P} \equiv 0$. Finally we can see that

$$D_t^k \nabla^{(l-1)} [D_t, \nabla] \mathcal{P} = D_t^k \nabla^{(l-1)} (\text{Rm} * \nabla \mathcal{P} + \mathcal{P} * \mathcal{S}),$$

and because $\mathcal{S} = (\nabla \text{Rm}) \circ \mathcal{P} = \nabla \mathcal{R} + \text{Rm} * \nabla \mathcal{P}$, $D_t^k \nabla^{(l-1)} [D_t, \nabla] \mathcal{P} \equiv 0$ at $t = 0$. This completes the proof. \square

4.2. Preservation of product structures. In the proof of Theorem 1.1, we will use the operator $\mathcal{F} : \Lambda^2 T^* M \rightarrow \Lambda^2 T^* M$ defined by

$$\mathcal{F}\omega(X, Y) = \omega(\hat{P}X, \check{P}Y) - \omega(\check{P}X, \hat{P}Y).$$

(See, for example, Section 2.2 of [10].) Observe that

$$\begin{aligned} \mathcal{P} \circ \mathcal{F}\omega(X, Y) &= \mathcal{F}\omega(\hat{P}X, \check{P}Y) + \mathcal{F}\omega(\check{P}X, \hat{P}Y) \\ &= \omega(\hat{P}^2X, \check{P}^2Y) - \omega(\check{P}\hat{P}X, \hat{P}\check{P}Y) + \omega(\hat{P}\check{P}X, \check{P}\hat{P}Y) - \omega(\check{P}^2X, \hat{P}^2Y) \\ &= \omega(\hat{P}X, \check{P}Y) - \omega(\check{P}X, \hat{P}Y). \end{aligned}$$

Therefore $\mathcal{P} \circ \mathcal{F} \equiv \mathcal{F}$.

Proof of Theorem 1. We have shown in Propositions 2.4 and 4.2 that the system \mathbf{X}, \mathbf{Y} satisfies the first two hypotheses of Theorem 3.1. Additionally, the curvature bounds from Proposition 2.6 imply that $|\mathbf{X}| \leq Ct^{-2}$. Thus, $\mathbf{X} \equiv 0$ and $\mathbf{Y} \equiv 0$ on $M \times [0, T]$. In particular, we know that $\mathcal{R} \equiv 0$ and $\nabla\mathcal{P} \equiv 0$ on $M \times [0, T]$.

We claim that $\nabla\hat{P} \equiv \nabla\check{P} \equiv 0$ and $\partial_t\hat{P} \equiv \partial_t\check{P} \equiv 0$. Similar to the proof of Lemma 7 in [10], if we define $W = \nabla\hat{P}$, then

$$D_t W_{ai}^j = [D_t, \nabla_a] \hat{P}_i^j = \nabla_p R_{pai}^c \hat{P}_c^j - \nabla_p R_{pab}^j \hat{P}_i^b + R_a^c W_{ci}^j.$$

Note that the first two terms combine to give

$$\begin{aligned} &\langle \nabla_{e_p} R(e_p, e_a) \check{P}e_i, \hat{P}e_j \rangle - \langle \nabla_{e_p} R(e_p, e_a) \hat{P}e_i, \check{P}e_j \rangle \\ &= \frac{1}{2} \left(\nabla_{e_p} \text{Rm}(\check{P}e_i^* \wedge \hat{P}e_j^*)(e_p, e_a) - \nabla_{e_p} \text{Rm}(\hat{P}e_i^* \wedge \check{P}e_j^*)(e_p, e_a) \right) \\ &= -\frac{1}{2} \nabla_{e_p} \text{Rm} \circ \mathcal{F}(e_i^* \wedge e_j^*)(e_p, e_a). \end{aligned}$$

But, since $\mathcal{P} \circ \mathcal{F} = \mathcal{F}$,

$$\nabla \text{Rm} \circ \mathcal{F} = \nabla \text{Rm} \circ \mathcal{P} \circ \mathcal{F} = \mathcal{S} \circ \mathcal{F} = 0,$$

so $D_t W_{ai}^j = R_a^c W_{ci}^j$. Thus, for any point $x \in M$, the function $f(t) = |\nabla\hat{P}|^2(x, t)$ satisfies

$$f'(t) \leq Cf$$

for some C depending on x . Since $f(0) = 0$, f is identically zero. Thus \hat{P} (and similarly \check{P}) remain parallel.

Hence, we have

$$R(\cdot, \cdot, \hat{P}(\cdot), \check{P}(\cdot)) = 0,$$

which implies that $\text{Rc} \circ \hat{P} = \hat{P} \circ \text{Rc}$ and $\text{Rc} \circ \check{P} = \check{P} \circ \text{Rc}$, and thus, from (2.1), $\partial_t \hat{P} = \partial_t \check{P} = 0$ on $[0, T]$. Theorem 1.1 follows. \square

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