We are mainly concerned with the completablity of unimodular rows to an invertible matrix. We first relate this to stable range of the ring. Further, we consider rows generating an idempotent of the ring to obtain completability under special conditions. We try to obtain analogues of important theorems regarding unimodular rows in the case of these “idempotent rows”.

By ring, we shall always mean a commutative ring with identity.

Section 1: Stable Range and Unimodular Elements

We consider a homomorphism from $\mathbb{R}^n$ to $\mathbb{R}$. The mapping may be represented as a row of length $n$. If we assume that the mapping is onto, we note that

1. The map must split.
2. If the row is $(a_1, ..., a_n)$, the elements $a_i$ must generate the unit ideal in $\mathbb{R}$. This is because the image of the epimorphism is the ideal generated by the $a_i$'s and it must contain 1. Let us call such rows “unimodular rows”.

Now consider the kernel of the epimorphism obtained from a unimodular row. Clearly the kernel is a projective module $P$ such that $P \oplus \mathbb{R} = \mathbb{R}^n$. We would like to obtain conditions under which $P$ is a free module. Since $\mathbb{R}$ is assumed to be commutative (and hence $\mathbb{R}$ has invariant basis number), this is equivalent to asking whether $P \approx \mathbb{R}^{n-1}$.

It is easy to see that the above condition is equivalent to the unimodular row being completable to an invertible matrix. For a proof see [7].

**Definition 1.1:** Stable Range: Let $\{a_1, ..., a_n\}$ be a set of elements generating the unit ideal. The set is said to be reducible if there exist $r_1, ..., r_{n-1}$ such that $a_1 + r_1 a_n, ..., a_{n-1} + r_{n-1} a_n$ still generate the unit ideal. If $m$ is such that every such sequence of length exceeding $m$ is reducible, the ring $\mathbb{R}$ is said to have stable range $m$. Let us denote “stable range” by $sr$.

**Definition 1.2:** Unimodular Elements: Let $M$ be an $\mathbb{R}$ module. Given any $x \in M$, we define the ideal $o_M(x) = \{ f(x) \mid f \in Hom(M, \mathbb{R}) \}$. $x$ is said to be a unimodular element if $o_M(x) = \mathbb{R}$. Therefore, by a unimodular row, we shall mean a row $a_1, ..., a_n$, the ideal generated by which is $\mathbb{R}$.

We shall first show that “long” unimodular rows over a ring are always completable.
Proposition 1.3: If a ring has stable range n, then every unimodular row of length > n is completable to an invertible matrix.

Proof: Let us consider the invertible matrix $I_m$, where $m > n$ and a unimodular row $(a_1, ..., a_m)$. Since the row is unimodular, there exist elements $r_1, ..., r_{m-1}$ such that $(a_1 + r_1a_m, ..., a_{m-1} + r_{m-1}a_m)$ is also a unimodular row. Let $b_i = a_i + r_ia_m$. Then, the following matrix is also invertible:

$$
\begin{pmatrix}
1 & b_1 & b_2 & \cdots & b_{m-1} \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}
$$

Since the sequence of $b_i$'s is unimodular, by means of elementary column operations we can change the 1 in the top row to $a_m$. Using this $a_m$, we can now sweep out the multiples $r_ia_m$ from the other columns and get $(a_m, a_1, a_2, ..., a_{m-1})$ as our top row. We have thus obtained a completion of the given unimodular sequence.

Corollary 1.4: Every unimodular row over a semilocal ring is completable.

Proof: This is because a semilocal ring has stable range 1. For proof, see [1].

We note that the row $(a_1, ..., a_n)$ is unimodular if and only if the ideal generated by the $a_i$'s is equal to $R$. Let the ideal generated by them be $I$. But $I = R$ if and only if $I_m = R_m$ for each maximal ideal $m$. Thus, the row is unimodular if and only if it is locally unimodular. We can use the method of localization to relate stable range to the dimension of the ring. However, we shall have to assume that the ring is noetherian.

Lemma 1.5: Let $R$ be a noetherian commutative ring and let $a_1, ..., a_n, s \in R$. Then there exist $c_1, ..., c_n \in R$ such that $ht(a_1 + sc_1, ..., a_n + sc_n)R_s \geq n$.

Proof: Let $p_1, ..., p_m$ be the minimal prime ideals of $R$ not containing $s$. Then, by prime avoidance, $a_1 + sR$ is not contained in $\cup p_i$. Thus, there exists $c_1$ such that $ht(a_1 + sc_1) \geq 1$.

Now, we consider $q_1, ..., q_k$, the minimal prime ideals of $R/(a_1 + sc_1)$ not containing $s$. Proceeding in the same fashion, we see that there exist $c_1, ..., c_n$ such that $ht(a_1 + sc_1, ..., a_n + sc_n) \geq n$. Also see [1].

Proposition 1.6: Let $R$ be a noetherian commutative ring and let $R$ have finite dimension. Then if $R$ is also reduced, $sr(R) \leq dim(R)$.

Proof: Let $a_1, ..., a_{n+1}$ be such that $n > dim(R)$. Now, we can find numbers $c_1, ..., c_n$ such that $ht(a_1 + a_{n+1}c_1, ..., a_n + a_{n+1}c_n)R_{a_{n+1}} \geq n$. Thus the height of the ideal in the ring $R_{a_{n+1}}$ also exceeds $dim(R)$. Consequently, it must be equal to $R$ itself, i.e. we have obtained a stable range kind of contraction (We require that the ring be reduced so that $R_{a_{n+1}}$ be always defined).

It is obvious that if any of the elements in the unimodular row is a unit or a zero, the vector becomes completable. Next, we try out a nilpotent element.

Proposition 1.7: Let $(a_1, ..., a_n)$ be a unimodular row such that $a_n$ is nilpotent. Then the row is completable to an invertible matrix.
Proof: We must have \( b_1, \ldots, b_n \) such that \( \sum a_i b_i = 1 \). But the element \( a_n b_n \) is nilpotent and hence \( a_1 b_1 + \ldots + a_{n-1} b_{n-1} = 1 \). Thus, the row \( a_1, \ldots, a_{n-1} \) is also unimodular. Consequently, the row \( a_1, \ldots, a_n \) is completable.

Corollary 1.8 : Let \( R \) be a commutative ring and let \( N \) denote the nilradical of \( R \). Then \( \text{sr}(R/N) = \text{sr}(R) \).

Proof: Let \( \text{sr}(R/N) = n \). Now suppose that there exist \( a_1, \ldots, a_{n+1} \in R \) which generate the unit ideal. Then, there exist \( r_1, \ldots, r_n \) such that \( a_1 + r_1 a_{n+1}, \ldots, a_n + r_n a_{n+1} \) generate \( R/N \). Thus this sum differs from 1 in \( R \) by a nilpotent element. Since units over \( N \) lift to units over \( R \), we are through.

Section 2: From Units to Idempotents

In this section we shall examine what happens if the rows \( a_1, \ldots, a_n \) generate the principal ideal generated by an idempotent instead of assuming that they are unimodular. We should get analogous results when we use idempotents instead of units.

We know that a unimodular row is an epimorphism from \( R^n \) to \( R \). The map splits and the kernel is a stably free projective module \( P \) of rank 1. This module \( P \) is free if and only if the unimodular row is completable to an invertible matrix.

In keeping with the notion of unimodular rows, we define idempotent rows as:

Definition 2.1 : Idempotent rows: Let \( a_1, \ldots, a_n \in R \) be such that the ideal generated by \((a_1, \ldots, a_n)\) equals the principal ideal generated by an idempotent. Then \((a_1, \ldots, a_n)\) is said to be an idempotent row.

We have the fact that \( P \oplus R \approx R^n \) and we want to know whether \( P \) is free. We shall start with an argument regarding the minimal number of generators for \( P \). This argument will use exterior powers:

Proposition 2.2: Let \( R \) be a commutative (and hence stably finite*) ring and let \( P \oplus R \approx R^n \). Then, if \( P \) is minimally generated by \( n-1 \) elements, \( P \) is free.

Proof: Let \( k \) be the minimal number of generators for \( P \). Then there exists \( Q \) such that \( P \oplus Q \approx R^k \). Clearly, \( k \leq n \). Let \( k < n - 1 \). Then we apply \( \wedge^n \) to both sides. We get \( \wedge^n P = 0 \) and also \( \wedge^{n-1} P = 0 \). However, if we apply \( \wedge^n \) to the relation \( P \oplus R \approx R^n \) and compare, we get a contradiction. Thus, \( k = n \) or \( k = n - 1 \). If \( k = n - 1 \), we add \( Q \) to both sides of the relation \( P \oplus R \approx R^n \) to get \( R^n \approx R^n \oplus Q \). Since a commutative ring is stably finite, we see that \( Q = 0 \) and hence \( P \) is free.

*(Definition: Stably finite: A ring \( R \) is said to stably finite if the matrix rings \( M_n(R) \) have the following property: if \( A, B \in M_n(R) \) are such that \( AB = I \), then \( BA = I \). This is equivalent to the following: if \( R^n \approx R^n \oplus Q \), then \( Q = 0 \).)

Now, we shall look at the stable ranges. If \( R \) has stable range \( n \), each of the rings \( eR \) and \( (1 - e)R \) also has finite stable range \((\leq n)\). Conversely, let
Let \( n = \max \{ sr(eR), sr((1 - e)R) \} \). Then, let us take a sequence \((a_1, ..., a_{n+1})\) of elements that generate \( R \). Clearly, \( ea_1, ..., ea_{n+1} \) generate \( eR \) and so also for \( 1 - e \). Thus, we find \( er_1, ..., er_n \) such that \((ea_1 + er_1, ..., ea_n + er_n) = (e)\) and similarly we get \((1 - e)s_i's\) for generating \((1 - e)\). The unit ideal is generated by the elements \( a_i + (r_i e + s_i(1 - e))a_{n+1} \). The stable range of the ring is, therefore, \( \leq n \).

Now, an idempotent row is a homomorphism from \( R^n \) to \( eR \). Let us fix the notation: \( e \) is an idempotent, \( I_1 = eR \) and \( I_2 = (1 - e)R \). The idempotent row is introduced when the mapping takes \( R^n \) not to the whole of \( R \), but only to a direct summand, in which case, the map must split. We obtain \( R^n = I_1 \oplus P \) for some projective module \( P \). The natural question to ask is: When is \( P = I_2 \oplus R' \) for some \( r' \)? The answer to this is obvious: there should exist elements \( b_1, ..., b_n \in I_2 \) such that \((a_1 + b_1, ..., a_n + b_n)\) is a completable unimodular row.

**Lemma 2.3** Let \((a_1, ..., a_n)\) be an idempotent row. Then the kernel of the associated homomorphism is of the form \( I_2 \oplus R' \) if and only if the idempotent row can be completed to a matrix \( A \) such that there exists a matrix \( B \) such that \( AB = \text{diag}(e, 1, 1, ..., 1) \).

**Proof:** Let \( b_1, ..., b_n \) be elements of \( I_2 \) such that \( \sum c_i(a_i + b_i) = 1 \). Let the row \((a_1 + b_1, ..., a_n + b_n)\) be completable to an invertible matrix. This is an if and only if condition for the kernel of the homomorphism represented by \((a_1, ..., a_n)\) to be of the form \( I_2 \oplus R' \). Consider the invertible matrix thus obtained, say \( A \), with right inverse \( B \). If the first row of \( A \) were to be changed to \((a_1, ..., a_n)\), then the first row of \( AB \) becomes the projection of the row \((1, 0, ..., 0)\) into \( I_1 \), i.e., \((e, 0, ..., 0)\) (Changing the first row of \( A \) from \((a_1 + b_1, ..., a_n + b_n)\) to \((a_1, ..., a_n)\) is equivalent to multiplying the elements of the first row of \( A \) by \( e \), since \( eb_i = 0 \) (since \( b_i \in I_2 \)). Hence the first row of \( AB \), which is \( I \) also gets multiplied by \( e \) and becomes \((e, 0, 0, ..., 0)\). Thus \( AB \) changes from \( \text{diag}(1, 1, ..., 1) \) to \( \text{diag}(e, 1, 1, ..., 1) \).

It is obvious that if a unimodular row \( a_1, ..., a_n \) is such that there exist \( a_1, ..., a_r \) \((r < n)\) such that \( a_1, ..., a_r \) is unimodular, we can complete the row to an invertible matrix. We shall now see that this is true even if the “subrow” is just an idempotent row.

**Proposition 2.4:** If \( x_1, ..., x_n \) is a unimodular row such that there exists \( r < n \) such that \( x_1, ..., x_r \) is an idempotent row, the unimodular row \( x_1, ..., x_n \) is completable.

**Proof:** Let \( x_1, ..., x_r \) be an idempotent row corresponding to an idempotent \( e \). If \( e = 1 \), we are through. Otherwise, let \( e \neq 1 \). Clearly, the row \((x_1, ..., x_r, 0, ..., 0)\) \((n - r)\)'s) is completable to a matrix \( A \) such that there exists a matrix \( B \) with \( AB = eI_n \) (with \( A, B \in M_n(eR) \)). We also note that \((1 - e)x_{r+1}, ..., (1 - e)x_n)\) is an idempotent row corresponding to \((1 - e)\). Similarly, \((0, ..., 0, (1 - e)x_{r+1}, ..., (1 - e)x_n)\) is completable to a matrix \( C \) such that there exists a matrix \( D \) with \( CD = (1 - e)I_n \) (with \( C, D \in M_n((1 - e)R) \)). Thus, \( A + B \) is an invertible matrix with inverse \( C + D \). The elements \( ex_{r+1}, ..., ex_n \) can be added to the last few columns by using the elements \( x_1, ..., x_r \) which generate \( eR \).
Corollary 2.5: If the unimodular row contains an idempotent, it is completable.

Corollary 2.6: If the unimodular row \((x_1, ..., x_n)\) contains \(i, j\) such that \(x_i \in eR\) and \(x_j \in (1 - e)R\), it is completable.

If \(P\) is an \(R\) module, for an element \(x\) of \(P\), let us define \(o(x) = \{ f(x) \forall f \in \text{Hom}(P, R) \}\). We know that \(o(x)\) is an ideal in \(R\). We are concerned with the situation in which \(o(x)\) is a direct summand of \(R\), i.e. \(o(x) = eR\) for some idempotent \(e\).

Proposition 2.7: Let \(R\) be a semi local ring and let \(P\) be an \(R\) module. Let \(\alpha, \beta\) be elements of \(P\) such that \(o(\alpha) = o(\beta) = eR\). Further, suppose that \((1 - e)\alpha = 0\). Then, there exists an automorphism \(\sigma\) of \(P\) such that \(\sigma(\alpha) = (1 - e, \alpha, 0)\) and \((1 - e, \beta, 0)\) respectively. Thus, there should exist an automorphism \(\sigma\) of \(I_2 \oplus P\) such that \(\sigma(1 - e, \alpha, 0) = (1 - e, \beta, 0)\). Further we know that \(\sigma\) will leave invariant all submodules containing both of the unimodular elements(This follows from a very important theorem of Bass, see [5]). Now, \(\sigma(1 - e, \alpha, 0) = \sigma(1 - e, 0, 0) + \sigma(0, \alpha, 0) = (1 - e, 0, 0) + (0, \beta, 0)\). We see that \(\alpha = e\alpha\) and \((1 - e)\alpha = 0\). Multiplying both sides by \(e\), we have \(\sigma(0, \alpha, 0) = (0, \beta, 0)\).

Section 3: Suslin’s Theorem

The completability of unimodular rows has great bearing upon the Serre Conjecture (proved independently by Quillen and Suslin). Serre’s Conjecture states that projective modules over all polynomial rings, i.e. rings of the form \(k[x_1, ..., x_n]\), \(k\) being a field, are free. The following theorem was one of the most important steps in Suslin’s proof of this conjecture. We shall concentrate on the method used by Suslin in proving the theorem below. We start by mentioning the theorem of Suslin:

Theorem 3.1: Let \(a_0, a_1, ..., a_n\) be a unimodular row. If the row \(a_0, a_1, ..., a_{n-1}\) is completable to an invertible matrix in the ring \(R/(a_n)\), then \(a_0, a_1, ..., a_{n-1}, a_n\) is completable to an invertible matrix in \(R\).

Proof: (Suslin[2]) We can easily see that given \(\alpha \in M_n(R)\) and \(\beta \in GL_n(R)\), the matrix
\[
\begin{pmatrix}
\alpha & 0 \\
0 & \beta
\end{pmatrix}
\]
can be changed to
\[
\begin{pmatrix}
\alpha \beta & 0 \\
0 & I_n
\end{pmatrix}
\]
by means of elementary transformations (Whitehead Lemma). (by elementary
transformation, we mean a pre or postmultiplication of the matrix by a matrix
of the form $I + a e_{ij}$ for $i \neq j$ where $a \in \mathbb{R}$ and $e_{ij}$ is the matrix having 1 in the
(i, j) position and 0’s elsewhere).

Let $a_0, a_1, \ldots, a_{n-1}$ be completable to an invertible matrix $\psi$ in $\mathbb{R}/a_n$. Then,
we have a $\phi$ such that $\psi \phi = I + a_n M$. We consider the matrix

$$A = \begin{pmatrix} \psi & a_n I_n \\ M & \phi \end{pmatrix}$$

$A$ is clearly invertible. Then we can use Whitehead’s Lemma to stack the powers
of $a_n$ appearing in the diagonal submatrix by means of elementary transforma-
tions. The matrix is then transformed to $B$, where $B$ has the form

$$B = \begin{pmatrix} \psi & a_n^n \\ M_1 & I_{n-1} \\ N_1 & 0 \end{pmatrix}$$

We use the $I_{n-1}$ submatrix to eliminate $n-1$ rows and $n-1$ columns, leaving
us with an invertible matrix having $a_0, a_1, \ldots, a_{n-1}, a_n^n$ as the first row.

We now adapt this method to prove:

**Proposition 3.2**: Let $(a_0, \ldots, a_{n-1})$ be completable in $\mathbb{R}/(a_n)$ to a matrix of
determinant $r$. Then, the row $(a_0, a_n, \ldots, a_{n-1}, a_n^n) \psi$ is completable(in $\mathbb{R}$) to a matrix
of determinant $r^n$.

**Proof**: Let the matrix in $\mathbb{R}/(a_n)$ be $\phi$. Consider the characteristic polynomial
of $\phi$. The characteristic polynomial yields a relationship of the form $\phi p(\phi) = r I$
in $\mathbb{R}/(a_n)$. When lifted to $\mathbb{R}$, we shall have an $M$ such that $\phi p(\phi) = r I + a_n M$.
Then, we consider the matrix

$$\psi = \begin{pmatrix} \phi & a_n I \\ M & p(\phi) \end{pmatrix}$$

The determinant of the above matrix is $r^n$. Applying Suslin’s procedure to
this matrix, we can obtain a $n \times n$ matrix with first row $a_1, \ldots, a_{n-1}, a_n^n$ and
determinant $r^n$.

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