CUBIC POITNS ON QUARTIC CURVES

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Abstract. In this paper, we prove a necessary condition when the equation $F(x^2, y^2, z^2) = 0$ has a solution in a cubic number field where $F(X, Y, Z)$ is a homogeneous quadratic form with rational coefficients. Then we use the necessary condition to study the family of the curve $x^4 + nx^2y^2 + y^4 = Dz^4$. Some compuational results are also given.

1. Introduction

This chapter studies the equation $F(x^2, y^2, z^2) = 0$, where $F(X, Y, Z)$ is a non-singular, irreducible, rational homogeneous quadratic polynomial in three variables. This equation defines a curve $\mathcal{C}$ of genus 3. The question of finding all rational points on $\mathcal{C}$ is interesting but there are currently no known algorithms to find all rational points on $\mathcal{C}$. We can ask if $\mathcal{C}$ has a point in an odd degree extension of $\mathbb{Q}$. Coray [6] showed that if $\mathcal{C}$ has a point in an odd degree extension of $\mathbb{Q}$ then $\mathcal{C}$ also has a point in $\mathbb{Q}$ or a cubic extension of $\mathbb{Q}$ (which we shall call a cubic point). Using algebraic number theory, Bremner, Lewis and Morton [2] gave some examples of the form $ax^4 + by^4 = cz^4$ which have no rational solutions but have cubic points where a, b, c are positive integers. Cassels [4] gave an algorithm to find cubic points in $\mathcal{C}$ and some examples where $\mathcal{C}$ has no cubic points. Using a different approach, Bremner [3] studied the equation $x^4 + y^4 = Dz^4$. Based on the techniques of Cassels [4] and Bremner [3], I will prove a necessary condition for $\mathcal{C}$ to have a cubic point, and then apply it to extend some old results on the equation $x^4 + nx^2y^2 + y^4 = Dz^4$.

2. Cubic points and their associated elliptic curves

We are interested in the genus 3 curve

$$\mathcal{C}\colon F(x^2, y^2, z^2) = 0.$$

We make the following assumption:

(2.1) if $X, Y, Z \in \mathbb{Q}$ such that $F(X, Y, Z) = 0$ and $XYZ = 0$ then $X = Y = Z = 0$

Consider three associated curves

$$\begin{cases} E_1\colon F(X, y^2, z^2) = 0 \\ E_2\colon F(x^2, Y, z^2) = 0 \\ E_3\colon F(x^2, y^2, Z) = 0. \end{cases}$$

By definition, a point $(x_0 : y_0 : z_0) \in \mathbb{P}^2(\bar{\mathbb{Q}})$ is a cubic point if $\mathbb{Q}(x_0 : y_0 : z_0)$ is a cubic number field. We need the following.

Lemma 2.1. *If $\mathcal{C}$ has a point in $\mathbb{P}_2(\mathbb{Q})$ then $\mathcal{C}$ also has a cubic point.*

Proof. See Cassels [4]. ☐

2010 *Mathematics Subject Classification.* Primary: 05C??. Secondary: 05C??

Key words and phrases. Diophantine equation, elliptic curve Chabauty.

Let $P = (\alpha, \beta, \gamma)$ be a cubic point on $\mathcal{C}$, and $G = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. G acts on P by

$$g(P) = (g(\alpha) : g(\beta) : g(\gamma)) \;\forall g \in G.$$

If $\gamma \neq 0$ then we can take $\gamma = 1$ and $P = (\alpha, \beta, 1)$. Without loss of generality, we can assume $\alpha \notin \mathbb{Q}$. Then $\beta = p(\gamma)$ where $p(x)$ is a polynomial of degree at most 2 with rational coefficients. The set of orbits of P is $\{(\alpha_i, p(\alpha_i), 1), i = 1, 2, 3\}$ where $\alpha_i, i = 1, 2, 3$ are all Galois conjugates of α. Each set of orbits of a cubic point on $\mathcal{C}$ is called a rational triplet.
Let T be a triplet $\{(\alpha_i, \beta_i, \gamma_i), i = 1, 2, 3\}$ on $\mathcal{C}$, then we have a triple of points $\{(\alpha_i^2, \beta_i, \gamma_i), i = 1, 2, 3\}$ on E_1. Because $\alpha_i^2, \beta_i^2, \gamma_i^2, \beta_i\gamma_i$ are linearly dependent over $\mathbb{Q}$, there are $r, s, t \in \mathbb{Q}$ such that

$$\alpha_i^2 = r\beta_i^2 + s\beta_i\gamma_i + t\gamma_i^2$$

for $i = 1, 2, 3$. This holds for $i = 1, 2, 3$ because the triple $\{(\alpha_i^2, \beta_i, \gamma_i), i = 1, 2, 3\}$ is invariant under G. The curve

$$X = ry^2 + syz + tz^2$$

meets E_1 at the triplet $\{(\alpha_i^2, \beta_i, \gamma_i), i = 1, 2, 3\}$ and a fourth point which is necessarily a rational point. Denote this point by $v_1(T)$. So v_1 maps each rational triplet T on $\mathcal{C}$ to a rational point $v_1(T)$ on E_1. See Cassels [4].
Similarly, we have maps v_2, v_3 from the set of rational triplets on $\mathcal{C}$ to the set of rational points on E_2, E_3 respectively. Thus for each rational triplet T on $\mathcal{C}$ we have a triple $(v_1(T), v_2(T), v_3(T)) \in E_1(\mathbb{Q}) \times E_2(\mathbb{Q}) \times E_3(\mathbb{Q})$.
Denote the groups of rational points on E_1, E_2, E_3 by $G_1, G_2,$ and G_3 respectively. Let $(P_1, P_2, P_3) \in G_1 \times G_2 \times G_3$. We want to find a rational triplet T such that

$$v_1(T) = P_1,\; v_2(T) = P_2\;, v_3(T) = P_3.$$

Cassels [4] and Bremner [3] showed that it is enough to find triplets T such that $v_i(T)$ is in the coset reprentatives of $G_i/2G_i$ for $i = 1, 2, 3$.
The map v_1 sends a rational triplet to a rational point on E_1; thus there is a non trivial rational point on E_1. So $F(X, Y, Z) = 0$ has non trivial solutions. Let

$$X : Y : Z = X(l, m) : Y(l, m) : Z(l, m) \tag{2.2}$$

be a parameterization of $F(X, Y, Z) = 0$ where $X(l, m), Y(l, m), Z(l, m)$ are degree 2 homogeneous polynomials in l, m with rational coefficients.
Because $P = (\alpha : \beta : \gamma)$ is a cubic point on $\mathcal{C}$, $(\alpha^2 : \beta^2 : \gamma^2)$ is a point on $F(X, Y, Z) = 0$. Let $(\alpha^2 : \beta^2 : \gamma^2)$ be parametrized by $\lambda : \mu$.
Let $f(x) = Ax^3 + Bx^2 + Cx + D$ be the defining polynomial of $\frac{\lambda}{\mu}$ with $A, B, C, D \in \mathbb{Z}$ and $\gcd(A, B, C, D) = 1$.
Let $\frac{\lambda_1}{\mu_1}, \frac{\lambda_2}{\mu_2}, \frac{\lambda_3}{\mu_3}$ be all conjugates of $\frac{\lambda}{\mu}$. Then $f(x)$ has the factorization

$$f(x) = A(x - \frac{\lambda_1}{\mu_1})(x - \frac{\lambda_2}{\mu_2})(x - \frac{\lambda_3}{\mu_3}).$$

Assume that $v_1(T) = (X_1, y_1, z_1)$ and let $(X_1 : y_1^2 : z_1^2)$ be parameterized by

$$X_1 : y_1^2 : z_1^2 = X(l_1, m_1) : Y(l_1, m_1) : Z(l_1, m_1) \tag{2.3}$$

where $l_1, m_1 \in \mathbb{Q}$.
Assume that in (2.2)

$$Z(l, m) = al^2 + blm + cm^2 \tag{2.4}$$

where $a, b, c \in \mathbb{Q}$.

The following lemma is due to Cassels [4]

Lemma 2.2. *Let* $d = y_1$ *then there are* $u, v, w, q \in \mathbb{Q}$ *and* $q \neq 0$ *such that*

$$\begin{cases} qA = m_1u^2 + 2aduw + gaw^2 \\ qB = -l_1u^2 + 2m_1uv + 2bduw + 2advw + (gb + ha)w^2 \\ qC = -2l_1uv + m_1v^2 + 2cduw + 2bdvw + (gc + hb)w^2 \\ qD = -l_1v^2 + 2cdvw + hcw^2 \end{cases} \tag{2.5}$$

where $y_1^2Z(l, m) - z_1^2Y(l, m) = (m_1l - l_1m)(gl + hm)$.

Proof. Lemma 2.1, Cassels [4]. □

Lemma 2.3. *If* $Z(l, m) = al^2 + cm^2$ *in* (2.4) *then*

$$Z(l_1, m_1)((a(cB - aD)^2 + c(cA - aC)^2) \in (\mathbb{Q}^*)^2$$

Proof. Substituting $b = 0$ in (2.5), we have

$$\begin{cases} qA = m_1u^2 + 2aduw + gaw^2 \\ qB = -l_1u^2 + 2m_1uv + 2advw + haw^2 \\ qC = -2l_1uv + m_1v^2 + 2cduw + gcw^2 \\ qD = -l_1v^2 + 2cdvw + hcw^2 \end{cases}$$

From the first and the third equations, we have

$$q(cA - aC) = m_1(cu^2 - av^2) + 2l_1auv.$$

From the second and the fourth equations, we have

$$q(cB - aD) = l_1(av^2 - cu^2) + 2m_1cuv.$$

Combining the above two equations, we have

$$\begin{aligned} q^2(c(cA - aC)^2 + a(cB - aD)^2) &= (cm_1^2 + al_1^2)(av^2 - cu^2)^2 + 4l_1^2a^2cu^2v^2 + 4m_1^2ac^2u^2v^2 \\ &= (cm_1^2 + al_1^2)((av^2 - cu^2)^2 + 4acu^2v^2) \\ &= (al_1^2 + cm_1^2)(av^2 + cu^2)^2. \end{aligned} \tag{2.6}$$

If $c(cA - aC)^2 + a(cB - aD)^2 = 0$ then $Z(cB - aD, cA - aC) = 0$.
Let $q = cA - aC$ and $p = cB - aD$ then $((X(p, q), Y(p, q), Z(p, q))$ is a solution of $F(X, Y, Z) = 0$ with $Z(p, q) = 0$. By (2.1), we have

$$X(p, q) = Y(p, q) = Z(p, q) = 0 \tag{2.7}$$

If $p \neq 0$ or $q \neq 0$, then from (2.7), $X(l, m), Y(l, m), Z(l, m)$ has a common factor $lq - mp$. Thus $F(X, Y, Z) = 0$ has a parameterization $(X_1(l, m) : Y_1(l, m) : Z_1(l, m))$ where $X_1(l, m), Y_1(l, m), Z_1(l, m)$ are linear polynomials in l, m. Therefore every point in $F(X, Y, Z) = 0$ is a rational point, which contradicts the existence of a cubic point, for example the point $(\alpha^2, \beta^2, \gamma^2)$.
Therefore

$$p = q = 0$$

So

$$cA - aC = cB - aD = 0$$

The polynomial $f(x)$ is now reducible with a factorization

$$f(x) = (Cx + D)(\frac{B}{D}x^2 + 1).$$

So

$$c(cA - aC)^2 + a(cB - aD)^2 \neq 0.$$

From (2.6), we have

$$Z(l_1, m_1)((a(cB - aD)^2 + c(cA - aC)^2) \in (\mathbb{Q}^*)^2$$

□

We consider the case where

$$Z(l, m) = al^2 + blm + cm^2 \quad \text{where} \quad a \neq 0 \quad \text{or} \quad c \neq 0. \tag{2.8}$$

By homogeneity, we assume that $a = 1$. Then

$$Z(l, m) = l^2 + blm + cm^2. \tag{2.9}$$

Let

$$\begin{cases} X_1(l, m) = X(l - \frac{b}{2}m, m) \\ Y_1(l, m) = Y(l - \frac{b}{2}m, m) \\ Z_1(l, m) = Z(l - \frac{b}{2}m, m) \end{cases}$$

Then

$$\begin{cases} X_1(l + \frac{b}{2}m, m) = X(l, m) \\ Y_1(l + \frac{b}{2}m, m) = Y(l, m) \\ Z_1(l + \frac{b}{2}m, m) = Z(l, m). \end{cases}$$

For example, because $Z(l, m) = l^2 + blm + cm^2$, we have $Z_1(l, m) = l^2 + (c - \frac{b^2}{4})m^2$. We have

$$F(X_1(l + \frac{b}{2}m, m), Y_1(l + \frac{b}{2}m, m), Z_1(l + \frac{b}{2}m, m)) = F(X(l, m), Y(l, m), Z(l, m)) = 0.$$

In other words, if $F(X, Y, Z) = 0$ has a parameterization $X(l, m) : Y(l, m) : Z(l, m)$ then it has a parametrization

$$X_1(l, m) : Y_1(l, m) : Z_1(l, m) = X(l - \frac{b}{2}m, m) : Y(l - \frac{b}{2}m, m) : Z(l - \frac{b}{2}m, m),$$

and conversely if $F(X, Y, Z) = 0$ has a parametrization $(X_1(l, m) : Y_1(l, m) : Z_1(l, m))$ then it also has a parametrization

$$X(l, m) : Y(l, m) : Z(l, m) = X_1(l + \frac{b}{2}m, m) : Y_1(l + \frac{b}{2}m, m) : Z_1(l + \frac{b}{2}m, m).$$

Because $(\alpha^2 : \beta^2 : \gamma^2)$ is a solution of $F(X, Y, Z) = 0$, $(\alpha^2 : \beta^2 : \gamma^2)$ has a parameterization

$$\alpha^2 : \beta^2 : \gamma^2 = X_1(L_1, M_1) : Y_1(L_1, M_1) : Z_1(L_1, M_1)$$

with

$$\frac{L_1}{M_1} = \frac{\lambda + \frac{b}{2}\mu}{\mu} = \frac{\lambda}{\mu} + \frac{b}{2}.$$

$\frac{\lambda}{\mu}$ is a root of $f(x) = Ax^3 + Bx^2 + Cx + D$, so $\frac{L_1}{M_1}$ is a root of

$$f(x - \frac{b}{2}) = Ax^3 + (-\frac{3}{2}Ab + B)x^2 + (\frac{3}{4}Ab^2 - Bb + C)x - \frac{1}{8}Ab^3 + \frac{1}{8}Bb^2 - \frac{1}{2}Cb + D$$

Let

$$\begin{cases} A_1 &= A \\ B_1 &= -\frac{3}{2}Ab + B \\ C_1 &= \frac{3}{4}Ab^2 - Bb + C \\ D_1 &= -\frac{1}{8}Ab^3 + \frac{1}{4}Bb^2 - \frac{1}{2}Cb + D \end{cases}$$

Now the intersection of the curve E_1 and the curve $X = ry^2 + syz + tz^2$ contains a rational point (X_1, y_1^2, z_1^2) satisfying

$$\begin{aligned} X_1 : y_1^2 : z_1^2 &= X(l_1, m_1) : Y(l_1, m_1) : Z(l_1, m_1) \\ &= X_1(l_1 + \frac{b}{2}m_1, m_1) : Y_1(l_1 + \frac{b}{2}m_1, m_1) : Z_1(l_1 + \frac{b}{2}m_1) \\ &= X_1(L, M) : Y_1(L, M) : Z_1(L, M) \end{aligned}$$

where $L = l_1 + \frac{b}{2}m_1$, and $M = m_1$.

Let $H_1(A, B, C, D) = (c_1B_1 - D_1)^2 + c_1(c_1A_1 - C_1)^2$ with $a_1 = 1$ and $c_1 = c - \frac{b^2}{4}$. Then

(2.10)

$$\begin{aligned} H_1(A, B, C, D) = &-b^3AD + b^2cAC + b^2BD - bc^2AB + 3bcAD - bcBC - bCD \\ &+ c^3A^2 - 2c^2AC + c^2B^2 - 2cBD + cC^2 + D^2. \end{aligned}$$

By applying Lemma 2.3 to $Z_1(l, m) = l^2 + (c - \frac{b^2}{4})m^2 = a_1l^2 + c_1m^2$, where $a_1 = 1$, and $c_1 = c - \frac{b^2}{4}$, we have

$$Z_1(L, M)H_1(A, B, C, D) \in (\mathbb{Q}^*)^2$$

Because $Z_1(L, M) = Z(l_1, m_1)$, we have

$$Z(l_1, m_1)H_1(A, B, C, D) \in (\mathbb{Q}^*)^2 \tag{2.11}$$

Lemma 2.4. *Assume that in* (2.2) $Z(l, m) = al^2 + blm + cm^2$. *Let*

$$\begin{aligned} H(A, B, C, D) = &-b^3AD + b^2cAC + ab^2BD - bc^2AB + 3abcAD - abcBC - a^2bCD \\ &+ c^3A^2 - 2ac^2AC + ac^2B^2 - 2a^2cBD + a^2cC^2 + a^3D^2 \end{aligned}$$

Then

$$Z(l_1, m_1)H(A, B, C, D) \in (\mathbb{Q}^*)^2$$

where (l_1, m_1) *is in* (2.3).

Proof. If $a \neq 0$ or $c \neq 0$ then we can assume that $a \neq 0$. By replacing b by $\frac{b}{a}$ and c by $\frac{c}{a}$ in (2.9) , (2.10) and (2.11), we have

$$Z(l_1, m_1)H(A, B, C, D) \in (\mathbb{Q}^*)^2.$$

If $a = c = 0$, then from the first and the fourth equations in (2.5), we have

$$\begin{cases} qA = m_1u^2 \\ qD = -l_1v^2. \end{cases} \tag{2.12}$$

$$\Rightarrow q^2AD = -m_1l_1(uv)^2.$$

Moreover, because $H(A, B, C, D) = -b^3AD$ and $Z(l_1, m_1) = bl_1m_1$, we have

$$q^2Z(l_1, m_1)H(A, B, C, D) = b^4(l_1m_1uv)^2 \tag{2.13}$$

From $a = c = 0$, we have $b \neq 0$. Because $f(x) = Ax^3 + Bx^2 + Cx + D$ has no linear factor over $\mathbb{Q}$, we have $A \neq 0$ and $D \neq 0$. In addition $q \neq 0$. So from (2.12), we have

$$m_1, l_1, u, v \neq 0$$

Therefore

$$q, u, v, l_1, m_1 \neq 0. \tag{2.14}$$

From (2.13) and (2.14), we have

$$Z(l_1, m_1)H(A, B, C, D) \in (\mathbb{Q}^*)^2.$$

□

On E_2, let $v_2(T) = (x_2, Y_2, z_2)$ be parameterized by

$$x_2^2 : Y_2 : z_2^2 = X(l_2, m_2) : Y(l_2, m_2) : Z(l_2, m_2)$$

where $l_2, m_2 \in \mathbb{Q}$.
On E_3, let $v_3(T) = (x_3, y_3, Z_3)$ be parameterized by

$$x_3^2 : y_3^2 : Z_3 = X(l_3, m_3) : Y(l_3, m_3) : Z(l_3, m_3)$$

where $l_3, m_2 \in \mathbb{Q}$.
From Lemma 2.4,

$$Z(l_1, m_1)H(A, B, C, D) \in (\mathbb{Q}^*)^2.$$

Hence $H(A, B, C, D) \in \mathbb{Q}^*$.
Similarly, we have

$$Z(l_2, m_2)H(A, B, C, D) \in (\mathbb{Q}^*)^2.$$

Therefore

$$Z(l_1, m_1)Z(l_2, m_2) \in (\mathbb{Q}^*)^2.$$

By symmetry, we have

$$X(l_2, m_2)X(l_3, m_3), \quad Y(l_1, m_1)Y(l_3, m_3) \in (\mathbb{Q}^*)^2.$$

Thus we have the following theorem

Theorem 2.5. *Let* $(X(l_i, m_i) : Y(l_i, m_i) : Z(l_i, m_i))$ *be a parameterization of* $v_i(T)$ *for* $i = 1, 2, 3$ *respectively. Then*

$$X(l_2, m_2)X(l_3, m_3),\ Y(l_1, m_1)Y(l_3, m_3), Z(l_1, m_1)Z(l_2, m_2) \in (\mathbb{Q}^*)^2$$

Remark 2.6. Bremner [3] proved Theorem 2.5 for the family of curves

$$x^4 + y^4 = Dz^4.$$

The approach in the paper is computational. The above proof of Theorem 2.5 takes a different approach and works for the general equation $F(x^2, y^2, z^2) = 0$.

3. Some applications

3.1. Equation $x^4 + y^4 = 4pz^4$.

Theorem 3.1. *Let p be an odd prime then the equation*

$$x^4 + y^4 = 4pz^4$$

does not have solutions in any odd degree number field except $xyz = 0$.

Proof. Consider the genus 3 curve

$$x^4 + y^4 = 4pz^4 \tag{3.1}$$

By Corollary 6.6, Coray [6], we only need to show (3.1) has no rational points or cubic points.
p is an odd prime; mod 2, (3.1) has no rational points. So we only need to show (3.1) has no cubic points.
Consider the curve

$$D_1\colon x^2 + y^4 = 4pz^4$$

Assume (3.1) has a non-trivial cubic point then D_1 has a non-trivial rational point. Let $(x_0, y_0, 1)$ be a rational point on the curve D_1 then the corresponding elliptic curve is

$$E_1\colon y^2 = x(x^2 + 16p) \tag{3.2}$$

Let r be the rank of E_1 over $\mathbb{Q}$.
If $r \leq 1$, then by Theorem 4, Bremner [3], C has no cubic points.
If $r \geq 2$, then by Proposition 6.2, Chapter X, Silverman [7]

$$r = 2 \text{ and } p \equiv 1 \mod 8.$$

A point on $x^2+y^4 = 4pz^4$ gives a point on $u^2+1 = 4pv^4$. By Proposition 6.5, Chapter X, Silverman [7], we have

$$(\frac{2}{p})_4 = 1$$

where $\left(\frac{\cdot}{\cdot}\right)_4$ denotes the bi-quadratic residue symbol.
Because $p \equiv 1 \mod 8$, there are A,B in $\mathbb{Z}^+$ such that

$$p = A^2 + B^2$$

where $2 \nmid A$ and $2|B$.
In addition, because $(\frac{2}{p})_4 = 1$, from Proposition 6.6, Chap X, Silverman [7], we have

$$AB \equiv 0 \mod 8$$

Therefore

$$8|B$$

Now, let (x, y, z) be a non trivial rational point in D_1. We can assume that $x, y, z \in \mathbb{Z}^+$ and $\gcd(x, y, z) = 1$. We have

$$x^2 + y^4 = 4pz^4.$$

Thus $2|x, y$. Let $x = 2s$, $y = 2t$ then

$$s^2 + 4t^4 = pz^4.$$

Because $4p = A^2 + B^2$, we have

$$(pz^2 + 2Bt^2)^2 = p(Bz^2 + 2t^2)^2 + A^2s^2$$

thus

$$(pz^2 + 2Bt^2 + As)(pz^2 + 2Bt^2 - As) = p(Bz^2 + 2t^2)^2.$$

We need the following lemma

Lemma 3.2. *(Silverman [7]) With the above notations, we have the following cases*
Case 1:

$$\begin{cases} pz^2 + 2Bt^2 + As = pu^2 \\ pz^2 + 2Bt^2 - As = v^2 \end{cases}$$

Case 2:

$$\begin{cases} pz^2 + 2Bt^2 + As = u^2 \\ pz^2 + 2Bt^2 - As = pv^2 \end{cases}$$

Case 3:

$$\begin{cases} pz^2 + 2Bt^2 + As = 2pu^2 \\ pz^2 + 2Bt^2 - As = 2v^2 \end{cases}$$

Case 4:

$$\begin{cases} pz^2 + 2Bt^2 + As = 2u^2 \\ pz^2 + 2Bt^2 - As = 2pv^2 \end{cases}.$$

Proof. We show that $\gcd(pz^2 + 2Bt^2 + As, pz^2 + 2Bt^2 - As)$ is either a square or 2 times a square.
Indeed, let $d = \gcd(pz^2 + 2Bt^2 + As, pz^2 + 2Bt^2 - As)$.
Let n be the square-free part of d. We want to show that $n = 1$ or $n = 2$.
We have

$$\det \begin{pmatrix} p & 2B & A \\ p & 2B & -A \\ B & 2 & 0 \end{pmatrix} = 4A(p - B^2) = 4A^3.$$

Thus $d|4A^3$; hence $n|4A^3$.
If $n > 1$ then let q be a prime divisor of n. We want to show that $q = 2$.
Assume that $q > 2$ then from $n|4A^3$, we have $q|A$.
Thus

$$s^2 = pz^4 - 4t^4 = (A^2 + B^2)z^4 - 4t^4 \equiv B^2z^4 - 4t^4 \equiv 0 \mod q$$

So $q|s$.
Let $v_q(d) = 2r + 1$, then $q^{2r+1}|Bz^2 + 2t^2$.
From

$$q^{2r+1}|pz^2 + 2Bt^2 + As = B^2z^2 + 2Bt^2 + A^2z^2 + As = B(Bz^2 + 2t^2) + A(Az^2 + s)$$

we have $q^{2r+1}|A(Az^2 + s)$. Because $q|s, q \nmid z$, we have $q^{2r+1}|A$.
If $v_q(Bz^2 + 2t^2) > 2r + 1$ then from $q|s, q^{2r+1}|A$, we have

$$q^{2r+2}| \gcd(pz^2 + 2Bt^2 + As, pz^2 + 2Bt^2 - As).$$

Thus $v_q(d) > 2r + 1$, a contradiction.
Therefore $v_q(Bz^2 + 2t^2) = 2r + 1$.
From $q > 2$, $\gcd(A, B) = 1$, $\gcd(s, z) = \gcd(s, t) = 1$ and $q|A, s$, we have $q \nmid Bz^2+2t^2$.
Therefore $q^{2r+1}||A^2z^4 + (Bz^2 + 2t^2)(Bz^2 - 2t^2) = s^2$, which is a contradiction.
So $n = 1$ or $n = 2$. □

Now if $4p = g^2 + h^2$, then the equation $X^2 + Y^2 = 4pZ^2$ has a parameterization

$$X : Y : Z = gl^2 - 2hlm - gm^2 : hl^2 + 2glm - hm^2 : l^2 + m^2$$

Point (x, y^2, z^2) in $X^2 + Y^2 = 4pZ^2$ is parameterized by a pair (l, m) satisfying

$$l : m = gx + hy^2 + Dz^2 : -hx + gy^2 = -hx + gy^2 : -gx - hy^2 + Dz^2$$

Let

$$\begin{cases} \alpha = gx + hy^2 + Dz^2 \\ \beta = -hx + gy^2 \\ \gamma = -gx - hy^2 + Dz^2 \end{cases}$$

then

$$\alpha\gamma = \beta^2$$

and

$$l : m = \alpha : \beta = \beta : \gamma$$

Thus

$$\begin{aligned} l^2 + m^2 &\equiv \alpha^2 + \beta^2 \mod (\mathbb{Q}^*)^2 \\ &\equiv \alpha^2 + \alpha\gamma \mod (\mathbb{Q}^*)^2 \\ &\equiv \alpha(\alpha + \gamma) \mod (\mathbb{Q}^*)^2 \\ &\equiv \alpha(2Dz^2) \mod (\mathbb{Q}^*)^2 \\ &\equiv 2p\alpha \mod (\mathbb{Q}^*)^2 \end{aligned} \tag{3.3}$$

Now, we have $p = A^2 + B^2$ where $8|B$.
Let $g = 2A, h = 2B$, $x = 2s$ and $y = 2t$ then

$$\alpha = gx + hy^2 + Dz^2 = 4(As + 2Bt^2 + pz^2)$$

Therefore,

$$l^2 + m^2 \equiv 2p\alpha \equiv 2p(As + 2Bt^2 + pz^2) \mod (\mathbb{Q}^*)^2 \tag{3.4}$$

Now $s^2 + 4t^4 = pz^4$ and $\gcd(x, y, z) = 1$, z and s are odd.
Consider Case 1 in Lemma 3.2

$$\begin{cases} pz^2 + 2Bt^2 + As = pu^2 \\ pz^2 + 2Bt^2 - As = v^2 \end{cases}$$

Taking modulo 8, we have

$$\begin{cases} 1 + As \equiv u^2 \mod 8 \\ 1 - As \equiv v^2 \mod 8 \end{cases}$$

A and s are odd, thus u, v are both even, thus $4|1 + As$ and $4|1 - AS$ which is impossible.
So Case 1 is impossible.
Similarly, Case 2 is impossible.
Case 3:

$$\begin{cases} pz^2 + 2Bt^2 + As = 2pu^2 \\ pz^2 + 2Bt^2 - As = 2v^2 \end{cases}$$

then from (3.4)

$$l^2 + m^2 \equiv 2p(As + 2Bt^2 + pz^2) \equiv 4p^2u^2 \equiv 1 \mod (\mathbb{Q}^*)^2$$

Case 4:

$$\begin{cases} pz^2+2Bt^2+As=2u^2 \\ pz^2+2Bt^2-As=2pv^2 \end{cases}$$

then from (3.4),

$$l^2+m^2 \equiv 2p(As+2Bt^2+pz^2) \equiv 4pu^2 \equiv p \mod (\mathbb{Q}^*)^2$$

So for the curve $D_1\colon x^2+y^4=4pz^4$, we have

$$l^2+m^2 \equiv 1 \text{ or } p \mod (\mathbb{Q}^*)^2$$

Now, consider the curve

$$D_2\colon x^4+y^2=4pz^4$$

We still have

$$4p=g^2+h^2$$

with $g=2A, h=2B$ and

$$p=A^2+B^2$$

with $B\equiv 0 \mod 8$.
In this case, because $x=2t, y=2s$, we have

$$4t^4+s^2=pz^4$$

Now the pair (l,m) satisfies

$$l:m=gx^2+hy+Dz^2:-hx^2+gy=-hx^2+gy:-gx^2-hy+Dz^2$$

By symmetry to the curve D_1, we also have

$$l^2+m^2 \equiv 2p(pz^2+2At^2+Bs) \mod (\mathbb{Q}^*)^2.$$

A similar argument shows that

$$pz^2+2At^2+Bs=pu^2 \text{ or } u^2$$

Thus

$$l^2+m^2\equiv 2 \quad \text{or} \quad 2p \mod (\mathbb{Q}^*)^2.$$

Now, for the curve

$$D_1\colon x^2+y^4=4pz^4$$

we get a pair (l_1,m_1) in which $l_1^2+m_2^2\equiv 1$ or $p \mod (\mathbb{Q}^*)^2$,
and for the curve

$$D_2\colon x^4+y^2=4pz^4$$

we get a pair (l_2,m_2) in which $l_2^2+m_2^2\equiv 2$ or $2p \mod (\mathbb{Q}^*)^2$.
Thus $(l_1^2+m_1^2)(l_2^2+m_2^2)\equiv 2$ or $2p \mod (\mathbb{Q}^*)^2$, hence is never a square.
So (3.1) has no non trivial points in any cubic extension of $\mathbb{Q}$. □

3.2. **Equation $x^4+nx^2y^2+y^4=Dz^4$.** This section studies the equation

$$x^4+nx^2y^2+y^4=Dz^4$$

Bremner [3] proved

Theorem 3.3. *Let D be a fourth power free integer such that D and $2D$ are not perfect squares. If the rank of the curve $x^2+y^4=Dz^4$ is at most one then the equation*

$$x^4+y^4=Dz^4$$

does not have any point in any cubic extension of $\mathbb{Q}$.

We prove the following theorem

Theorem 3.4. *Let n, D be non-zero integers such that D is fourth power free, $2-n, (2+n)D, (4-n^2)D,$ and D are not perfect squares. Assume that the rank of the curve $x^2+nxy^2+y^4=Dz^4$ is at most one. Then the equation*

$$x^4+nx^2y^2+y^4=Dz^4 \tag{3.5}$$

does not have any nontrivial solution in any odd degree extension of $\mathbb{Q}$ except $xyz=0$. In particular, the equation $x^4+nx^2y^2+y^4=Dz^4$ has no rational solutions except $x=y=z=0$.

Proof. Consider the curve

$$C\colon x^4+nx^2y^2+y^4=Dz^4$$

By Corollary 6.6, Coray [6], if C has a non-trivial point in an odd degree extension of $\mathbb{Q}$ then C has a non trivial rational point or a cubic point. By Lemma 2.1, we only need to show that C has no cubic points.
Because $n^2-4\notin\mathbb{Z}^2$, the equation $X^2+nXY+Y^2=DZ^2$ has no rational solution (X,Y,Z) with $XYZ=0$ except $X=Y=Z=0$; therefore the condition (2.1) is satisfied.
Assume that C has a non-trivial cubic point. Then the curve

$$E_1\colon X^2+nXy^2+y^4=Dz^4$$

has a non trivial rational point. There are $g,h\in\mathbb{Q}^*$ such that $D=g^2+ngh^2+h^4$. The equation

$$X^2+nXY+Y^2=DZ^2 \tag{3.6}$$

has a parameterization
(3.7)
$$X:Y:Z=(g+nh^2)l^2+2h^2lm-gm^2 : -h^2l^2+2glm+(ng+h^2)m^2 : l^2+nlm+m^2$$

where

$$l:m=X+gZ:Y+h^2Z \tag{3.8}$$

Let $A=1-\frac{n^2}{4}$ and $(a,b)=(g+\frac{n}{2}h^2,h)$.

Lemma 3.5. *The curve $C_1\colon X^2+Ay^4=Dz^4$ has the elliptic curve model*

$$E\colon v^2=u(u^2+4AD)$$

via the following maps $\phi\colon C_1 \to E$ *with* $\phi(X,y,z) = (u,v)$ *where*

$$\begin{cases} u = \frac{2(Dz^2-b^2y^2A+aX)}{(bz-y)^2} \\ v = \frac{4(aDz^3+DXz-b^3Xy-aby^3A)}{(bz-y)^3} \end{cases}$$

and $\psi\colon E \to C_1$ *with* $\psi(u,v) = (X,y,z)$ *where*

$$\begin{cases} X = a^3u^3 - 12ab^2ADu^2 - 4a^3ADu + 8bAD(D+Ab^4)v - 16ab^2A^2D^2 \\ y = abv - 2uD + 4ADb^2 \\ z = -2Ab^3 + av - 4bAD. \end{cases}$$

Proof. By using Magma [1], we can check that ϕ and ψ are inverses of each other and

$$\begin{cases} \phi(a,b,1) = (0:1:0) \\ \phi(-a,b,1) = (\frac{4ADb^2}{a^2}, \frac{-4AD(a^2+2Ab^4)b}{a^3}) \\ \phi(a,-b,1) = (\frac{a^2}{b^2}, \frac{a(a^2+2Ab^4)}{b^3}) \\ \phi(-a,-b,1) = (0,0). \end{cases} \tag{3.9}$$

□

We need the following

Lemma 3.6. *Let d be a non-zero integer such that $d \neq 4$ and $-d$ is not a rational square. Then the group of torsion points on $y^2 = x(x^2+d)$ is $\{(0,0),(0,1,0)\}$.*

Proof. Prop 6.1, Chapter X, Silverman [7]. □

Because $4AD \neq 4$ and $-4AD = (n^2-4)D$ is not a square, by Lemma 3.6, the torsion subgroup of E is $\mathbb{Z}/2\mathbb{Z}$ and is generated by $(0,0)$. So if the rank of E_1 is 0, then there are only finitely many points on E_1; thus there are only also many finitely many points on C_1 via the map

$$\begin{cases} \zeta\colon E_1 \to C_1 \\ \zeta(x,y,z) = (x + \frac{n}{2}y^2, y, z). \end{cases} \tag{3.10}$$

The only torsion points on E are $(0,0)$ and $(0:1:0)$; therefore C_1 has only 2 rational points, but C_1 has at least 4 points $(\pm a, \pm b, 1)$. So if the rank of E_1 is 0, then E_1 has no rational points except $(0,0,0)$. Therefore C has no point in any cubic extension of $\mathbb{Q}$.

Now consider the case when the rank of E_1 is 1. Then the ranks of both C_1 and E are one.

Two curves

$$\begin{cases} E_1\colon x^2 + nxy^2 + y^4 = Dz^4 \\ E_2\colon x^4 + nx^2y + y^2 = Dz^4 \end{cases}$$

have rank 1.

A rational triplet T on C gives a pair $(v_1(T), v_2(T))$ on $E_1(\mathbb{Q}) \times E_2(\mathbb{Q})$.

By following Bremner [3], Cassels [4], we only need to find T such that $v_i(T)$ is in the set of the coset reprentatives of $E_i(\mathbb{Q})/2E_i(\mathbb{Q})$ for $i = 1,2$.

Point $\phi(-a,b,1) = (\frac{4ADa^2}{b^2}, \frac{a(a^2+2Ab^4)}{b^3})$ is of infinite order because the only non-zero torsion point on E is $(0,0)$. We also have $\psi(-a,b,1)$ is not divisible by 2 because if $\psi(-a,b,1) = 2(u_0,v_0)$ then

$$\frac{4ADb^2}{a^2} = \frac{(4AD-u_0)^2}{(2v_0)^2}$$

which is impossible because $AD = (1-n^2/4)D$ is not a square. Therefore

$$E(\mathbb{Q}) = \mathbb{Z}\times\mathbb{Z}/2\mathbb{Z} \text{ and } E(\mathbb{Q})/2E(\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z},$$

so the coset representatives of $E(\mathbb{Q})/2E(\mathbb{Q})$ are $(0:1:0)$ and $\psi(-a,b,1)$.
From (3.9) and (3.10), we have

$$\begin{cases}(\phi\circ\zeta)^{-1}(0,1,0) = \zeta^{-1}(\phi^{-1}(0,1,0)) = \zeta^{-1}(a,b,1) = (a-\frac{n}{2}b^2,b,1) = (g,h,1)\\ (\phi\circ\zeta)^{-1}(0,0,1) = \zeta^{-1}(\phi^{-1}(0,1,0)) = \zeta^{-1}(-a,b,1) = (a-\frac{n}{2}b^2,b,1) = (-g-nh^2,h,1)\end{cases}$$

So the pull backs of $(0:1:0)$ and $(0,0,1)$ on E_1 are $(g,h,1)$ and $(-g-nh^2,h,1)$. Thus we only need to find triplet T such that

$$v_1(T) \in \{(g,h,1),(-g-nh^2,h,1)\} \tag{3.11}$$

Similarly, on E_2 we only need to consider triplet T such that

$$v_2(T) \in \{(b,a-\frac{n}{2}b^2,b,1),(b,-a-\frac{n}{2}b^2,1)\} = \{(h,g,1),(h,-g-nh^2,1)\} \tag{3.12}$$

The point $(g,h,1)$ on E_1 corresponds to the point $(g:h^2:1)$ on $X^2+NXY+Y^2 = DZ^2$. From (3.8), $(g:h^2:1)$ is parameterized by

$$l_1 : m_1 = (g+g):(h^2+h^2) = g:h^2.$$

Similarly, point $(-g-nh^2,h,1)$ on E_1 corresponds to point $(-g-nh^2,h^2,1)$ on (3.6). From (3.8), $(-g-nh^2:h:1)$ is parameterized by

$$l_1 : m_1 = (-g-nh^2+g):(h^2+h^2) = -n:2$$

Because

$$Z(g,h^2) = D \text{ and } Z(-n,2) = 4-n^2,$$

we have

$$Z(l_1,m_1) \mod (\mathbb{Q}^*)^2 \in \{D,\ 4-n^2.\} \tag{3.13}$$

Similarly, points $(h,g,1)$, and $(h,-g-nh^2,1)$ on E_2 correspond to points $(h^2,g,1)$ and $(h^2,-g-nh^2,1)$ on (3.6) which are parameterized by

$$l_2 : m_2 \in \{1:1,\ h^2+g:-g-(n-1)h^2\}.$$

Because

$$\begin{cases}Z(1,1) = n+2\\ Z(h^2+g,-g-(n-1)h^2) = (2-n)(g^2+ngh^2+h^4) = (2-n)D,\end{cases},$$

we have

$$Z(l_2,m_2) \mod (\mathbb{Q}^*)^2 \in \{n+2,(2-n)D\} \tag{3.14}$$

From (3.13) and (3.14), we have

$$Z(l_1,m_1)Z(l_2,m_2) \mod (\mathbb{Q}^*)^2 \in \{(n+2)D, 2-n, (n+2)(4-n^2), (2-n)(4-n^2)D\}.$$

Because $(n+2)(4-n^2) = (2-n)(2+n)^2$ and $(2-n)(4-n^2) = (2-n)^2(2+n)$, we have

$$Z(l_1,m_1)Z(l_2,m_2) \mod (\mathbb{Q}^*)^2 \in \{(n+2)D, 2-n\}.$$

By the assumption on n, D then $(n+2)D$, $2-n$ are not perfect squares. Therefore, $Z(l_1, m_1)Z(l_2, m_2) \notin (\mathbb{Q}^*)^2$, which contradicts Theorem 2.5. $\square$

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4. Appendix

We give some solutions to the equation $x^4 + y^4 = Dz^4$ where $D < 10000$. The strategy to find solutions is to search for cubic number fields $\mathbb{Q}(t)$ where t satisfies $at^3 + bt^2 + ct + d = 0$ in which the equation $x^4 + y^4 = Dz^4$ has a solution. This comes down to search for rational points on some homogeneous polynomials in a, b, c, d of degree 64. The algorithm was also stated in Cassels [4] and Bremner [3]. We have the following table

Table 1: Solutions of $x^4 + y^4 = Dz^4$

D	Cubic polynomial defining t	x	y	z
443	$t^3 - t^2 + 8$	$t^2 + 1$	$t - 2$	1
577	$t^3 - t^2 - 60972t - 13813000$	$\frac{1941t^2-284099t-61950438}{22858226}$	$\frac{499t^2-43596t+17636618}{11429113}$	1
1217	$t^3 + 719887t + 738159912$	$\frac{727t^2-11403t-938217784}{244185550}$	$\frac{-1217t^2+1362613t-35940086}{244185550}$	1
1777	$-\frac{5320687}{602320}t^3 + \frac{1241317}{602320}t^2 + \frac{2597617}{602320}t + 1$	$\frac{1}{53098}(-888723t^2 + 1558403t + 117662)$	$\frac{1}{191}(-764t^2 + 5102t + 788)$	$t^2 + 1$
1873	$-\frac{532076238522349807}{435082757960482167 4}t^3 + \frac{868069163214966164}{4350827579604821674}t^2 - \frac{5030676841258403279}{4350827579604821674}t + 1$	$-\frac{606437152469148}{103858567341425}t^2 - \frac{128177543406547}{103858567341425}t + \frac{529523869513673}{103858567341425}$	$-\frac{149980402}{82089865}t^2 + \frac{1608906352}{82089865}t - \frac{2946041588}{82089865}$	$t^2 + 1$
1889	$\frac{3404641469214788113836}{7601169982050224925503377}t^3 + \frac{3739298627236469973839401}{3040467992820089970201350 8}t^2 - \frac{5358841821246864376216483}{7601169982050224925503377}t + 1$	$-\frac{346527660909507688}{77363288673972050}t^2 + \frac{1695621331731884835 7}{77363288673972050}t - \frac{44941517494280046162}{77363288673972050}$	$-\frac{3508429512}{604901640}t^2 - \frac{1943705195747}{604901640}t + \frac{5514042765442}{604901640}$	$t^2 + 1$
2753	$\frac{1432652495664}{40800233234177}t^3 + \frac{32707381153769}{40800233234177}t^2 + \frac{59802385215158}{40800233234177}t + 1$	$-\frac{112040214}{12653550}t^2 - \frac{437879971}{12653550}t - \frac{31092181}{12653550}$	$-\frac{572936}{204620}t^2 - \frac{5041589}{204620}t - \frac{6361759}{204620}$	$t^2 + 1$
2801	$-\frac{784719925160}{5214612456061}t^3 + \frac{6219693961004}{5214612456061}t^2 + \frac{11669679283964}{5214612456061}t + 1$	$-\frac{14760380}{11613750}t^2 - \frac{200447838}{11613750}t - \frac{27372889}{11613750}$	$-\frac{5086835}{455835}t^2 + \frac{15189576}{455835}t + \frac{15663158}{455835}$	$t^2 + 1$
3137	$-\frac{519948}{96557}t^3 + \frac{949533}{96557}t^2 - \frac{278842}{96557}t + 1$	$-\frac{3199956}{137160}t^2 + \frac{4088495}{137160}t - \frac{826609}{137160}$	$-\frac{5402}{254}t^2 + \frac{4301}{254}t + \frac{179}{254}$	$t^2 + 1$
3229	$\frac{1575576}{2137801}t^3 + \frac{23094361}{2137801}t^2 + \frac{13652618}{2137801}t + 1$	$-\frac{718}{1283}t^2 + \frac{127006}{1283}t + \frac{38350}{1283}$	$-\frac{180}{36}t^2 - \frac{4751}{36}t - \frac{1451}{36}$	$t^2 + 1$

3649	$-\frac{155738}{23409}t^3+\frac{261473}{23409}t^2-\frac{3508}{1377}t+1$	$-\frac{13334}{1275}t^2+\frac{35251}{1275}t-\frac{4641}{1275}$	$-\frac{15373}{675}t^2+\frac{11447}{675}t+\frac{648}{675}$	t^2+1
4001	$\frac{49472722}{29393679}t^3+\frac{12826189}{1799613}t^2+\frac{43710244}{9797893}t+1$	$-\frac{950274}{56433}t^2-\frac{1795724}{56433}t-\frac{366516}{56433}$	$-\frac{3549}{507}t^2-\frac{23807}{507}t-\frac{8694}{507}$	t^2+1
4993	$\frac{1362540805930698631469349 6}{666093567914829449321251595 3}t^3+\frac{6217520526390261465182872 16}{666093567914829449321251595 3}t^2-\frac{4255745377389670505965077888}{6660935679148294493212515953}t+1$	$-\frac{53377836168590873 94}{813010109053557850}t^2+\frac{754411007184061994 34}{813010109053557850}t-\frac{1161597962820634985 31}{813010109053557850}$	$-\frac{346777873328}{177348640510}t^2-\frac{26045264170212}{177348640510}t+\frac{81613818831463}{177348640510}$	t^2+1
6353	$-\frac{368030872}{674441021}t^3+\frac{1558384512}{674441021}t^2-\frac{1774757484}{674441021}t+1$	$-\frac{2683308664}{82123770}t^2+\frac{7103733136}{82123770}t-\frac{4114661791}{82123770}$	$-\frac{14446986}{483650}t^2+\frac{30633054}{483650}t-\frac{10160939}{483650}$	t^2+1
6481	$-\frac{110440}{7569}t^3+\frac{15801}{841}t^2-\frac{15238}{7569}t+1$	$-\frac{527}{20}t^2+\frac{31983}{800}t-\frac{2393}{800}$	$-3t^2+\frac{387}{20}t+\frac{63}{20}$	t^2+1
7522	$\frac{176102191524634056 25}{346526594016943898921}t^3+\frac{173786227542143492 6875}{346526594016943898921}t^2-\frac{13883338747574140 6719}{31502417637903990811}t+1$	$-\frac{10145220196827}{2307626943028}t^2+\frac{300937570118302}{2307626943028}t-\frac{131724269039407}{2307626943028}$	$-\frac{6282377092855625}{647248491480800}t^2-\frac{47530814249370002}{647248491480800}t+\frac{21414917567843063}{647248491480800}$	t^2+1
7537	$-\frac{478793115}{140014628}t^3+\frac{877971339}{70007314}t^2+\frac{852368421}{140014628}t+1$	$-\frac{152981}{7738}t^2+\frac{363105}{7738}t+\frac{80442}{7738}$	$-\frac{6390}{1065}t^2+\frac{60906}{1065}t+\frac{15466}{1065}$	t^2+1
8882	$-\frac{125318930782936719922501719961 07}{857209049554969488421624727333 17}t^3+\frac{455254862340237140417945864625 83}{857209049554969488421624727333 17}t^2-\frac{775403571441021980099814111907 93}{857209049554969488421624727333 17}t+1$	$-\frac{120389005864572635737212 7}{159902332875541342451838}t^2+\frac{366957684224509975216477 8}{159902332875541342451838}t-\frac{846666671828542465486121 9}{159902332875541342451838}$	$-\frac{900259362351500941}{141938281994105982}t^2-\frac{1326247943951450118}{141938281994105982}t-\frac{332102859432140333}{141938281994105982}$	t^2+1
9281	$-\frac{123119}{4682}t^3+\frac{158196}{2341}t^2-\frac{74015}{4682}t+1$	$\frac{1}{29}(-5654t^2+14435t-1735)$	$\frac{1}{23}(-92t^2+t-30)$	t^2+1
9596	$-\frac{18086271804 99}{25009857744533}t^3+-\frac{844612424797}{25009857744533}t^2+\frac{18094138210859}{25009857744533}t+1$	$-\frac{344224782153}{54286633458}t^2+\frac{743427722104}{54286633458}t+\frac{178784756189}{54286633458}$	$-\frac{11977557}{736922}t^2+\frac{1682696}{736922}t+\frac{44517537}{736922}$	t^2+1
9649	$\frac{2204102626544}{19142972461433}t^3+\frac{15989802592176}{19142972461433}t^2+\frac{1595414088193}{19142972461433}t+1$	$-\frac{4122916}{437990}t^2+\frac{624389}{437990}t-\frac{2085339}{437990}$	$-\frac{13075520107}{2371559425}t^2+\frac{9334335848}{2371559425}t-\frac{28125830693}{2371559425}$	t^2+1

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