Functional Equations

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1 Introduction

We would like to use the following property of polynomials:
Let $P(x) = a_o + a_1x + ... + a_nx^2$, where $a_0, ..., a_n \in \mathbb{R}(\mathbb{Q}, \mathbb{Z}, ...)$. If

$$P(x) = 0$$

for infinitely many x, then $a_0 = ... = a_n = 0$.
This simple property helps us solve many hard functional equation problems.

2 Basic problems

1. Find all function $f : (0, \infty) \to \mathbb{R}$ such that

$$f(x + y) = f(x) + f(y) + A$$

$$f(x^n) = f(x)^n + B$$

for all $x, y \in (0, \infty)$, and A, B are real constant.

Solution
Assume that f is a function satisfying the hypothesis.
Let $p \in \mathbb{N}$ and $p > 0$, then for all $m \in \mathbb{Z}^+$, we have

$$f(mp) = mf(p) + (m - 1)A$$

Now for all $x > 0$ and $m \in \mathbb{Z}^+$, from 1 we have

$$f((x + mp)^n) = f(x + mp)^n + B$$

$$
\begin{aligned}
f((x+pm)^n) &= f(\sum_{k=0}^{n}\binom{n}{k}(pm)^k x^{n-k}) \\
&= \sum_{k=0}^{n-1}\binom{n}{k}(pm)^k f(x^{n-k}) + m^n p^{n-1} f(p) + (\sum_{k=0}^{n-1}\binom{n}{k}(pm)^k - 1)A + (m^n p^{n-1} - 1)A
\end{aligned}
$$

$$
\begin{aligned}
f(x+mp)^n &= (f(x) + m(f(p)+A))^n + B \\
&= \sum_{k=0}^{n}\binom{n}{k}(m(f(p)+A))^k f(x)^{n-k} + B
\end{aligned}
$$

Case 1: $A \neq 0$.
The coefficient of m^{n-1} in $f((x+pm)^m)$ is $np^{n-1}f(x) + np^{n-1}A$.
The coefficient of m^{n-1} in $f(x+mp)^n$ is $n(f(p)+A)^{n-1}f(x)$.
Therefore,

$$np^{n-1}f(x) + np^{n-1}A = n(f(p)+A)^{n-1}f(x)$$

or

$$((f(p)+A)^{n-1} - p^{n-1})f(x) = p^{n-1}A$$

Because $A \neq 0$, we have $f(x)$ is a constant function, hence $f(x) = -A$ and $-A = (-A)^n + B$.
Case 2: $A = 0$. The coefficient of m^{n-2} in $f((x+mp)^n)$ is $\binom{n}{2}p^{n-2}f(x^2)$.
The coefficient of m^{n-2} in $f(x+mp)^n$ is $\binom{n}{2}f(p)^{n-2}f(x)^2$.
Thus

$$p^{n-2}f(x^2) = f(p)^{n-2}f(x)^2$$

So for all $x > a$

$$f(x^2) = cf(x)^2$$

where $c = (\frac{f(p)}{p})^{n-2}$.
f is also additive, so $f = ax$.
If $a \neq 0$ then $B = A = 0$ and $f(x) = ax$ with $a^n = a$.

2. Let $a > 0$, $A, B \in \mathbb{R}$. Find all function $f : (a, \infty) \to \mathbb{R}$ such that

$$f(x+y) = f(x) + f(y) + A$$

$$f(x^n) = f(x)^n + B$$

Solution Similar to Problem 1, just fix $p \in \mathbb{N}$ and $p > a$. Then we have 2 cases:
Case 1: $A \neq 0$ then the only solution is a constant function.
Case 2: $A = 0$ then we will have $f(x^2) = cf(x)^2$ where c is a constant for all $x > a$. This implies $f(x) \geq 0$ for all $x > a^2$ or $f(x \leq 0)$ for all $x > a^2$.

Fix $b > a^2$ then $f(rb) = rf(b)$ for all $\frac{a^2}{b} < r \in \mathbb{Q}$.
If $c > 0$ then $f(x) \geq 0$ for all $x > a^2$.
For $x > a$, let $x = ub$.
Pick a sequence $\{u_k\}_{k=1,2,..}$ of rational numbers increasing to u with $u_1 b > a$.
Fix k and choose $M_k \in$ large enough such that $M_k(u - u_k)b > a^2$ then

$$M_k(f(ub) - f(u_k b)) = f(M_k(u - u_k)b) \geq 0$$

So $f(x) = f(ub) \geq f(u_k b) = u_k f(b)$
Taking the limit we get $f(x) \geq uf(b)$.
Similarly, by picking a sequence of rational numbers decreasing to u, we have $f(x) \leq uf(b)$.
Therefore, $f(x) = uf(b) = \frac{f(b)}{b}x$ for all $x > a$.
So $f(x) = \alpha x$ for some $\alpha \in \mathbb{R}$.
This forces $B = 0$ and $\alpha^n = \alpha$ If $c < 0$, we also get $f(x) = \beta x$ for some $\beta \in \mathbb{R}$.

3. Let $a > 0$, $n \in \mathbb{Z}$, $n > 1$. $A, B \in \mathbb{R}$, $A > 0$. Find all function $f : (a, \infty) \to \mathbb{R}$ such that

$$f(x + y) - f(x) - f(y) = \text{constant}$$

$$f(Ax^n) = Bf(x)^n + \text{constant}$$

Solution Let $f(x + y) - f(x) - f(y) = \alpha$ and $f(Ax^n) - Bf(x)^n = \beta$ with $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$.
Fix $p \in$ and $p > a$. Then again, for all $m \in \mathbb{Z}^+$, we have

$$Bf(x + mp)^n = B(f(x) + mf(p) + m\alpha)^n = B(f(x) + mb)^n$$

and

$$\begin{aligned} f(A(x + mp)^n) &= f(A(\sum_{k=0}^{n-1} \binom{n}{k} (mp)^k x^{n-k}) + (mp)^n) \\ &= \sum_{k=0}^{n-1} \binom{n}{k} (mp)^k f(Ax^{n-k}) + m^n p^{n-1} f(Ap) + \\ &+ (\sum_{k=0}^{n-1} \binom{n}{k} (mp)^k + m^n p^{n-1} - 1)\alpha \end{aligned}$$

Comparing the coefficients of m^n, we have

$$Bb^n = p^{n-1} f(Ap)$$

Comparing the coefficients of m^{n-1}, we have

$$b^{n-1}Bf(x) = p^{n-1}(f(Ax) + \alpha)$$

Comparing the coefficients of m^{n-2}, we have

$$b^{n-2}Bf(x)^2 = p^{n-2}f(Ax^2) + p^{n-2}\alpha$$

If $b = f(p) + \alpha = 0$, then

$$p^{n-2}(f(Ax^2) + \alpha) = 0$$

for all $x > a$, thus $f(x) = -\alpha$ for all $x > Aa^2, x > a$.
Now fix $y > a$, then pick x large enough, we will have

$$f(x + y) = -\alpha$$

and $f(x) = -\alpha$ hence $f(y) = -\alpha$ Therefore $f(x) = -\alpha$ for all $x > a$.
So $-\alpha = B(-\alpha)^n + \beta$.
If $b = f(p) + \alpha \neq 0$, then

$$p^{n-1}f(Ap) = Bb^n \neq 0$$

hence $f(Ap) \neq 0$.
We have

$$p^{n-1}f(Ap)f(x) = Bb^n f(x) = bp^{n-1}(f(Ax) + \alpha)$$

Hence $f(Ap)f(x) = f(Ax) + \alpha$ Now again, for all $y > a$ then

$$f(x + my) = f(x) + m(f(y) + \alpha)$$

$$f(A(x + my)) = f(Ax) + m(f(Ay) + \alpha)$$

Therefore, we have

$$f(Ap)(f(x) + m(f(y) + \alpha)) = f(Ax) + m(f(Ay) + \alpha) + \alpha$$

This holds for every $m \in \mathbb{Z}^+$, so

$$f(Ap)(f(y) + \alpha) = f(Ay) + \alpha$$

But

$$f(Ap)f(y) = f(Ay) + \alpha$$

so

$$f(Ap)\alpha = 0$$

Because $f(Ap) \neq 0$, we have $\alpha = 0$.
Therefore

$$f(Ax^2) = B(\frac{b}{p})^{n-2} f(x)^2$$

for all $x > a$.
This shows that $f(x) = 0$ for all $x > a$ or $f(x) \leq 0$ for all $x > a$ or $f(x) \geq 0$ for all $x > a$.
By similar argument to the previous problem, we have $f(x) = cx$ for some real constant c.

4. Let $a > 0$, $m, n \in^*$. Find all function $f : (a, \infty) \to \mathbb{R}$ such that

$$f(x+y) = f(x) + f(y)$$
$$f(x^{n+m}) = x^n f(x^m)$$

for all $x \in (a, \infty)$.

Solution

Let $k = n+m$ Let f be a function satisfying the problem. Fix $p \in$ and $p > a$.
For all $r \in^*$, we have

$$f(rp) = rf(p)$$

Now

$$\begin{aligned} f((x+rp)^k) &= f((rp)^k + \sum_{i=0}^{k-1} \binom{k}{i} (rp)^i x^{k-i}) \\ &= r^k p^{k-1} f(p) + \sum_{i=0}^{k-1} \binom{k}{i} (rp)^i f(x^{k-i}) \end{aligned}$$

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Regard $f((x+rpk)^k)$ as a polynomial in r, then the coefficient of r^{k-1} is $kp^{k-1}f(x)$

$$(x+rp)^n f((x+rp)^m) = (\sum_{i=0}^{n} \binom{n}{i} (rp)^i x^{n-i})(\sum_{i=0}^{m-1} \binom{m}{i} (rp)^i f(x^{m-i}) + r^m p^{m-1} f(p))$$

Regard $(x+rp)^n f((x+rp)^m)$ as a polynomial in r, then the coefficient of $r^{n+m-1} = r^{k-1}$ is $p^{k-1}mf(x) + p^{k-2}nxf(p)$. Now from

$$f((x+rp)^k) = (x+rp)^n f((x+rp)^m)$$

for all $r \in^*$, we must have

$$kp^{k-1}f(x) = p^{k-1}mf(x) + p^{k-2}nxf(p)$$

for all $x > a$. So $f(x) = cx$ for all $x > a$ and c is a constant.

5. Let $a > 0$, $m, n \in^*$, find all function $f : (a, \infty) \to \mathbb{R}$ such that

$$f(x + y) = f(x) + f(y)$$

$$f(x^{n+m}) = x^n f(x)^m$$

for all $x \in (a, \infty)$.
Solution
Let $k = m + n$.
Similar the Problem 2, just fix $p > a$ and $p \in^*$. Regard $f((x + rpk)^k)$ as a polynomial in r, then the coefficient of r^{k-1} is $kp^{k-1}f(x)$.
Regard $(x + rp)^n f((x + rp))^m$ as a polynomial in r, then the coefficient of $r^{n+m-1} = r^{k-1}$ is $p^n f(p)^{m-1} m f(x) + p^{n-1} n x f(p)^n$. So we have

$$(m + n)p^{m+n-1} f(x) = mp^n f(p)^{m-1} f(x) + np^{n-1} f(p)^n x$$

or

$$(m + n)p^m f(x) = pmf(p)^{m-1} f(x) + nf(p)^n x$$

or

$$p((m + n)p^{m-1} - mf(p)^{m-1})f(x) = nf(p)^n x$$

If $f(p) = 0$ then $f(x) = 0$ for all $x > a$.
If $f(p) \neq 0$ then we have $f(x) = cx$ for all $x > a$ and some constant $c \in \mathbb{R}$ satisfying $c = c^m$.
So $f(x) = 0$ or $f(x) = cx$ with $c^{m-1} = 1$.

6. Let $a > 0$, $m, n \in^*$,$A > 0$, $B \in \mathbb{R}$, find all function $f : (a, \infty) \to \mathbb{R}$ such that

$$f(x + y) - f(x) - f(y) = \text{constant}$$

and

$$f(Ax^n) - Bx^n f(x^m) = \text{constant}$$

for all $x, y > a$.
Solution Solution uses the same method.

7. Let $p, q \in \mathbb{Z}^+$, $a > 0$. $A, B \in \mathbb{R}$. Find all function $f : (a, \infty) \to \mathbb{R}$ such that

$$f(x + y) = f(x) + f(y) + A$$

$$f(x^p) = \frac{f(x)^{p+q}}{x^q} + B$$

for all $x, y > a$.
Solution Let $D = (a, \infty)$.
As before, fix $r > a$, $r \in$.
We have $f(mr) = mf(r)$ for all $m \in \mathbb{Z}^+$.
We have

$$f(x)^{p+q} = x^q f(x^p) - Bx^q$$

For all $m \in \mathbb{Z}^+$ then

$$f(x + mr) = f(x) + m(f(r) + A) = f(x) + mb$$

with $b = f(r) + A$.
Thus

$$f(x + mr)^{p+q} = (f(x) + mb)^{p+q} = \sum_{k=0}^{p+q} \binom{p+q}{k} m^k (b^k f(x)^{p+q-k})$$

as a polynomial in m, then the coefficient of m^{p+q-1} is $(p+q)b^{p+q-1}f(x)$ On the other hand,

$$\begin{aligned} f((x + mr)^p) &= f(\sum_{k=0}^{p-1} \binom{p}{k})(mr)^k x^{p-k} + (mr)^p) \\ &= \sum_{k=0}^{p-1} \binom{p}{k} (mr)^k f(x^{p-k}) + m^p r^{p-1} f(r) + (m^p r^{p-1} + \sum_{k=0}^{p-1} \binom{p}{k} (mr)^k - 1)A \end{aligned}$$

Thus

$$\begin{aligned} (x + mr)^q (f((x + mr)^p) - B) = (x + mr)^q (\sum_{k=0}^{p-1} \binom{p}{k} (mr)^k f(x^{p-k}) + m^p r^{p-1} f(r) + (m^p r^{p-1} \\ + \sum_{k=0}^{p-1} \binom{p}{k} (mr)^k - 1)A) \end{aligned}$$

As a polynomial in m, the coefficient of m^{p+q-1} is

$$r^q p r^{p-1}(f(x) + A) + qxr^{p+q-2} f(r)$$

So we have

$$(p + q)(f(r) + A)^{p+q-1} f(x) = pr^{p+q-1}(f(x) + A) + qxr^{p+q-2} f(r)$$

Thus

$$\alpha f(x) = \beta x$$

where $\alpha = (p+q)(A+f(r))^{p+q-1} - pr^{p+q-1}$ and $\beta = qxr^{p+q-2}f(r)$.
If $\exists r \in, r > a$ with $f(r) \neq 0$ then we have $f(x) = cx$ for all $x > a$ for some real constant c, this forces $A = B = 0$.
If $f(r) = 0$ for all $r \in$ with $r > a$, then we have

$$\alpha = (p+q)A^{p+q-1}$$

and

$$\beta = 0$$

so $(p+q)A^{p+q-1}f(x) = 0$ for all $x > a$. If $A \neq 0$ then $f(x) = 0$ for all $x > a$, so $B = 0$.
If $A = 0$ then we have

$$f(x+y) = f(x) + f(y)$$

for all $x, y > a$.
then

$$f(x+mr) = f(x) + mf(r) = f(x)$$

Thus

$$f(x+mr)^{p+q} = f(x)^{p+q}$$

$$\begin{aligned} f((x+mr)^p) &= f(\sum_{k=0}^{p} \binom{p}{k} (mr)^k x^{p-k}) \\ &= \sum_{k=0}^{p-1} \binom{p}{k} (mr)^k f(x^{p-k}) + f((mr)^p) \\ &= \sum_{k=0}^{p-1} \binom{p}{k} (mr)^k f(x^{p-k}) \end{aligned}$$

So we have

$$(x+mr)^q (\sum_{k=0}^{p-1} \binom{p}{k} (mr)^k f(x^{p-k}) - B) = f(x)^{p+q}$$

for all $x > a$. The right hand side if a polynomial in m, so the leading coefficient is 0, hence $r^q \binom{p}{p-1} r^{p-1} f(x) = 0$, hence $f(x) = 0$.
In conclusion, we have the following cases
Case 1: $f(x) = cx$ for some constant $c \in R$, this forces $A = B = 0$.
Case 2: $f(x) = c$ for some constant $c \in \mathbb{R}$, this forces $c = B = 0$ and hence $A = 0$.

8. Let $a > 0$, $m, n \in^*$,$A > 0$, $B \in \mathbb{R}$, find all function $f : (a, \infty) \to \mathbb{R}$ such that

$$f(x + y) - f(x) - f(y) = \text{constant}$$

and

$$f(Ax^n) - B\frac{f(x^{m+n})}{x^m} = \text{constant}$$

for all $x, y > a$.

3 Some applications

9. (Romanian 2009) Find all function $f : \mathbb{R} \to R$ such that

$$f(x^3 + y^3) = xf(x^2) + yf(y^2)$$

for all $x, y \in \mathbb{R}$.
This is a special case of the following problem.

10. Let $a > 0$, $m, n \in \mathbb{Z}^+$. Find all function $f : (a, \infty) \to \mathbb{R}$ such that

$$f(x^{n+m} + y^{n+m}) = x^n f(x^m) + y^n f(y^m)$$

11. (China TST ??) Let $n \in \mathbb{Z}$, $n > 1$. Find all function $f : \mathbb{R} \to R$ such that

$$f(x^n + f(y)) = f(x)^n + y$$

for all $x, y \in \mathbb{R}$.

12. Let $a > 0$, $A \in \mathbb{R}$, $n \in \mathbb{Z}$ and $n \geq 2$. Find all function $f : (a, \infty) \to (a - a^n, \infty)$ such that

$$f(x^n + f(y)) = f(x)^n + y + A$$

13. Let $a > 0$, $m, n \in \mathbb{Z}^+$, $A \in \mathbb{R}$. Find all function $f : (a, \infty) \to (a - a^n, \infty)$ such that

$$f(x^{n+m} + f(y)) = x^n f(x^m) + y + A$$

14. Let $a > 0$, $m, n \in \mathbb{Z}^+$. $S = \{n \in : s > a\}$. Find all function $f : S \to \mathbb{R}$ such that

$$f(x^{n+m} + y^{n+m}) = x^n f(x^m) + y^n f(y^m)$$

15. Let $a > 0$, $m, n \in \mathbb{Z}^+$. $S = \{n \in: s > a\}$. Find all function $f : S \to \mathbb{R}$ such that

$$f(x^{n+m} + f(y)) = x^n f(x^m) + y$$

16. Let $a > 0$, $m, n \in \mathbb{Z}^+$. Find all function $f : (a, \infty) \to \mathbb{R}$ such that

$$f(x^m + y^m + z^m) = \frac{f(x^{n+m})}{x^m} + \frac{f(y^{n+m})}{y^m} + \frac{f(z^{m+n})}{z^m}$$

17. Let $a > 0$, $m, n \in \mathbb{Z}^+$. $S = \{n \in: s > a\}$. Find all function $f : S \to \mathbb{Z}^+$ such that

$$f(x^n + f(y)) = \frac{f(x^{n+m})}{x^m} + y$$

18. Let $m, n \in \mathbb{Z}^+$. Find all function $f : (0, \infty) \to \mathbb{R}$ such that

$$f(\frac{1}{x^m} + \frac{1}{y^m} + \frac{1}{z^m}) = \frac{f(x^m)}{x^{n+m}} + \frac{f(y^n)}{y^{n+m}} + \frac{f(z^n)}{z^{n+m}}$$

19. Let $p, q, r \in \mathbb{Z}^+$ and $a \in \mathbb{R}^+$. Find all function $f : (a, \infty) \to \mathbb{R}$ such that

$$f(x^p + y^q + z^r) = \frac{f(x)^{p+q}}{x^q} + \frac{f(y)^{q+r}}{y^r} + \frac{f(z)^{r+p}}{z^p}$$

Solution Assume that f is such a function. Fix $b > a$.
Then let $y = z = b$, we have

$$f(x^p + A_1) = \frac{f(x)^{p+q}}{x^q} + A$$

where $A_1 = b^q + b^r$ and $A = \frac{f(b)^{q+r}}{b^r} + \frac{f(b)^{r+p}}{b^p}$.
Similarly, let $B_1 = b^r + b^p, C_1 = b^p + b^q$ and $B = \frac{f(b)^{r+p}}{b^p} + \frac{f(b)^{p+q}}{b^q}$, $C = \frac{f(b)^{p+q}}{b^q} + \frac{f(b)^{q+r}}{b^r}$. Then we have

$$f(y^q + B_1) = \frac{f(y)^{q+r}}{y^r} + B$$

$$f(z^r + C_1) = \frac{f(z)^{r+p}}{z^p} + C$$

So we have

$$f(x^p + y^q + z^r) = f(x^p + A_1) + f(y^q + B_1) + f(z^r + C_1) + D$$

where $D = -A - B - C$ for all $x, y, z > a$.
Let $T > a^p, a^q, a^r$ then

$$f(x + y + z) = f(x + A_1) + f(y + B_1) + f(z + C_1)$$

for all $x, y, z > T$.
Fix $z > T$, then for all $x > T + A1$ and $y > T$, we have

$$f(x+A_1)+f(y+B_1)+f(z+C_1) = f(x+y+z) = f(x-A1+y+A_1+z) = f(x)+f(y+A_1+B_1)+f(z+C$$

Therefore

$$f(x) - f(x - A_1) = f(y + A_1 + B_1) - f(y + B_1)$$

for all $x > A_1 + T$ and $y > T$. So for all $x, y > T + A_1, T + B_1$ then $f(x + A_1) - f(x) = f(y + A_1) - f(y)$ so for all $x > T + A_1$ then $f(x + A_1) = f(x) + D_1$ for some $C_1 \in \mathbb{R}$.
Now fix $\alpha \in R$ with $\alpha > T + A_1, T + B_1, T + C_1, 1$ then by a similar argument as above, we have for all $y, z > \alpha$ then

$$f(y + B_1) = f(y) + D_2$$

$$f(z + C_1) = f(z) + D_3$$

for some real constant D_2, D_3.
Then we will have

$$f(x + y + z) = f(x) + f(y) + f(z) + E$$

for all $x, y, z > \alpha$ and $E = D + D_1 + D_2 + D_3$.
Also, for $\alpha > 1$ then if $x > \alpha$ then $x^p\alpha$, hence

$$f(x^p + A_1) = f(x^p) + D_1$$

Therefore,

$$f(x^p) = \frac{f(x)^{p+q}}{x^q} + F$$

with $F = A - D_1$
Now we go back to a basic problem: find all function $f : (\alpha, \infty) \to \mathbb{R}$ such that

$$f(x + y + z) = f(x) + f(y) + f(z) + E$$

and

$$f(x^p) = \frac{f(x)^{p+q}}{x^q} + F$$

Now replace z by $z + t$ in $f(x + y + z + t)$ we have

$$f(x + y + z + t) = f(x + y) + f(z) + f(t) + E = f(x) + f(y) + f(z + t) + E$$

for all $x, y, z, t > \alpha$.
Therefore

$$f(x+y) - f(x) - f(y) = f(z+t) - f(z) - f(t)$$

for all $x, y, z, t > \alpha$.
so

$$f(x+y) - f(x) - f(y) = G$$

for all $x, y > \alpha$ and G is a real constant.
Then we have

$$f(x+y) - f(x) - f(y) = \text{constant}$$

and

$$f(x^p) - \frac{f(x)^{p+q}}{x^q} = \text{constant}$$

for all $x, y > \alpha$. So the only possible functions are

$$f(x) = cx$$

for some $c \in \mathbb{R}$.
Plug in the original equation

$$f(x^p + y^q + z^r) = \frac{f(x)^{p+q}}{x^q} + \frac{f(y)^{q+r}}{y^r} + \frac{f(z)^{r+p}}{z^p}$$

we have $cx^p + cy^q + cz^r = c^{p+q}x^q + c^{q+r}y^q + c^{r+p}z^r$ for all $x, y, z > a$.
Hence $c = c^{p+q} = c^{q+r} = c^{r+p}$.
If $c \neq 0$ then $c^p = c^q = c^r = 1$,so $c = 1$.
So the only possible functions are $f(x) = 0$ for all $x > a$ or $f(x) = x$ for all $x > a$.

20. Let $p, q, r \in \mathbb{Z}^+$ and $a \in \mathbb{R}^+$, $A \in \mathbb{R}$. Find all function $f : (a, \infty) \to \mathbb{R}$ such that

$$f(x^p + y^q + z^r) = \frac{f(x)^{p+q}}{x^q} + \frac{f(y)^{q+r}}{y^r} + \frac{f(z)^{r+p}}{z^p} + A$$

Solution Solve this problem in the same way as the previous problem, then
If $A \neq 0$, there is no function.
If $A = 0$ then $f(x) = 0$ for all $x > a$ or $f(x) = x$ for all $x > a$.

21. Let $A, B, C \in \mathbb{Z}^+$, $a, b, c, u \in \mathbb{R}^+$, $\alpha, p, q, r \in \mathbb{R}$. Find all function $f : (u, \infty) \to \mathbb{R}$ such that

$$f(ax^A + by^B + cz^C) = pf(x)^A + qf(y)^B + rf(z)^C + \alpha$$

Solution As before, just fix some $v > u$. Then let $x = y = z = v$ respectively, we will have for all $x, y, z > u$ then

$$f(ax^A + D_1) = pf(x)^A + E_1$$

$$f(by^B + D_2) = qf(y)^B + E_2$$
$$f(cz^C + D_3) = rf(z)^C + E_3$$

where $D_1, D_2, D_3 > 0$ and $E_2, E_2, E_3 \in \mathbb{R}$.
so from the original equation, we will have

$$f(ax^A + by^B + cz^C) = f(ax^A + D_1) + f(bx^B + D_2) + f(cx^C + D_3) + E_4$$

where $E_4 = -E_1 - E_2 - E_3$ (the values of E_4 does not matter much).
Just take $w > 0$ large enough then from the above equation, we have

$$f(x + y + z) = f(x + D_1) + f(y + D_2) + f(z + D_3)$$

for all $x, y, z > w$.
Now using the standard trick by writing

$$\begin{aligned} x + y + z + D_1 &= x + (y + D_1) + z \\ &= (x + D_1) + y \end{aligned}$$

we have

$$f(x+D_1)+f(y+D_1+D_2)+f(z+D_3) = f(x+2D_1)+f(y+D_2)+f(z+D_3)$$

for all $x, y, z > w$.
Hence

$$f(x + 2D_1) - f(x + D_1) = f(y + D_1 + D_2) - f(y + D_2)$$

for all $x, y > w$.
Now just replace w by

$$w_1 = w + D_1 + D_2 + D_2$$

, then

$$f(x + D_1) - f(x) = f(y + D_1) - f(y)$$

for all $x, y > w_1$.
Therefore,

$$f(x + D_1) - f(x) = \text{constant}$$

for all $x > w_1$.
Similarly,

$$f(y + D_2) - f(y) = \text{constant}$$

$$f(z + D_3) - f(z) = \text{constant}$$

for all $y, z > w_1$.
Thus

$$f(x + y + z) = f(x) + f(y) + f(z) + E$$

for all $x, y, z > w_1$ and E is a real constant.
Now just replace $z = x_1 + y_1$ for $x_1, y_1 > w_1$ then

$$\begin{aligned} f(x + y + x_1 + y_1) &= f(x) + f(y) + f(x_1 + y_1)) + E \\ &= f(x + y) + f(x_1) + f(y_1) + E \end{aligned}$$

for all $x, y, x_1, y_1 > w_1$.
Therefore,

$$f(x + y) - f(x) - f(y) = \text{constant}$$

for all $x, y > w_1$.
Now be enlarging w_1, we will have $ax^A > w_1$ for all $x > w_1$, thus we will have

$$f(ax^A + D_1) = f(ax^A) + \text{constant}$$

for all $x > w_1$.
Therefore,

$$f(ax^A) = pf(x)^A + \text{constant}$$

We go back a basic problem: Let $a, u > 0, A \in \mathbb{Z}^+$, $p \in \mathbb{R}$. Find all function $f : (u, \infty) \to \mathbb{R}$ such that

$$f(x + y) - f(x) - f(y) = \text{constant}$$

and

$$f(ax^A) = pf(x)^A + \text{constant}$$

22. Let $p, q, r \in \mathbb{Z}^+$ and $a, A, B, C \in \mathbb{R}^+$, and $A_1, B_2, C_2\alpha \in \mathbb{R}$. Find all function $f : (a, \infty) \to \mathbb{R}$ such that

$$f(Ax^p + By^q + Cz^r) = A_1 f(x)^p + B_1 f(y)^q + C_1 f(z)^r + \alpha$$

for all $x, y, z > a$.
Special case:
1,Find all function $f : (2019, \infty) \to \mathbb{R}$ such that

$$f(x^3 + 2y^5 + 3z^7) = f(x)^3 - 2f(y)^5 + 3f(z)^7$$

23. Let $p, q, r \in \mathbb{Z}^+$ and $a, A, B, C \in \mathbb{R}^+$, and $A_1, B_2, C_2\alpha \in \mathbb{R}$. Find all function $f : (a, \infty) \to \mathbb{R}$ such that

$$f(Ax^p + By^q - Cz^r) = A_1 f(x)^p + B_1 f(y)^q + C_1 f(z)^r + \alpha$$

for all $x, y, z > a$ such that $Ax^B y^q - Cz^r > a$
Solution Just fix $z_0 > a$, and we can assume that $x, y > M > a$ where M is large enough.
We will have

$$f(Ax^p + a_1) = A_1 f(x)^p + b_1$$

$$f(By^q + a_2) = B_1 f(y)^q + b_2$$

then we will have $f(x + y - Cz_0^r) = f(x + a_1) + f(y + b_1) + D$ with D is a constant (notice that z_0 is a constant).
Then by the usual trick, write $x+y$ as $x+y = x-a_1+y+a_1 = x+b_1+y-b_1$ then we will have

$$f(x + a_1) = f(x) + \text{constant}$$

and

$$f(y + b_1) = f(y) + \text{constant}$$

So then we have

$$f(x + y - E) = f(x) + f(y) + \text{constant}$$

Replace x by $x + E$ and y by $y + E$ respectively, we will have

$$f(x + E) + f(y) = f(x) + f(y + E)$$

hence

$$f(x + E) = f(x) + \text{constant}$$

Therefore, we again have

$$f(x + y) = f(x) + f(y) + \text{constant}$$

and

$$f(Ax^p) = A_1 f(x)^p + \text{constant}$$

From our basic problem, the only possible solution is $f(x) = \alpha x$ for some $\alpha \in \mathbb{R}$.

24. Find all function $f : (2019, \infty) \to \mathbb{R}$ such that

$$f(2x^3 - 3y^4 + 4z^5) = 2f(x)^3 - 3f(y)^4 + 4f(z)^5$$

for all $x, y, z > 2019$ and $2x^3 - 3y^4 + 4y^5 > 2019$.
Solution Just fix $y = a > 2019$ then we have

$$f(2x^3 + 4z^5 - A) = 2f(x)^3 + 4f(z)^5 - B$$

for all $x, y > b$ with b large enough, then

$$f(2x^3 + \text{constant}) = 2f(x^3) + \text{constant}$$

and

$$f(4z^5 + \text{constant}) = 4f(z)^5 + \text{constant}$$

Hence

$$f(2x^3 + 4z^5 - A) = f(2x^3 + \text{constant}) + f(4z^5 + \text{constant}) + \text{constant}$$

So we go back to a basis problem:

$$f(x + z - A) = f(x + B) + f(z + C) + D$$

and

$$f(2x^3 + B) = 2f(x)^3 + E$$

where A, B, C, D, E are constant.
just replace z by $z + A$, we have

$$f(x + z) = f(x + B) + f(z + A + C) + D$$

so we only have to look at

$$f(x + z) = f(x + B) + f(z + C) + D$$

and

$$f(2x^3 + B) = 2f(x)^3 + E$$

Using the usual trick,

$$f(x) + f(z + B + C) + D = f(x - B + z + B) = f(x + B) + f(z + C) + D$$

hence

$$f(x + B) - f(x) = f(z + B + C) - f(z + C)$$

so

$$f(x+B) - f(x) = \text{constant}$$

hence

$$f(2x^3+B) = f(2x^3) + \text{constant}$$

Similar

$$f(z+C) - f(z) = \text{constant}$$

This show that

$$f(x+z) = f(x) + f(z) + \text{constant}$$

and

$$f(2x^3) = 2f(x)^3 + \text{constant}$$

Thus

$$f(x) = cx + d$$

so

$$c(2x^3) + d = 2(cx+d)^3 + \text{constant}$$

$c = 0$ then $f(x)$ is a constant function and

$$d = 2d^3 - 3d^4 + 4d^5$$

$c \neq 0$ then comparing the coefficients of x^2 forces $d = 0$ and $c = \pm 1$.
Plug in the original equation, we get $c = 1$,so $f(x) = x$ or $f(x) = \text{constant}$ for x large enough.
Now just fix $x > 2019$, and pick y, z large enough then $2x^3 - 3y^4 + 4z^5$ is large enough, hence we can compute

$$2f(x)^3 = f(2x^3 - 3y^4 + 4z^5) + 3f(y)^4 - 4f(z)^5$$

So $f(x) = x$ for all $x > 2019$ or $f(x) = \text{constant}$ for all $x > 2019$.

25. Find all function $f : (2019, \infty) \to \mathbb{R}$ such that

$$f(2x^2 - 3y^3 + 6z^4) = 2f(x^2) - 3f(y)^3 + 6f(z)^4$$

for all $x, y, z > 2019$ such that $2x^2 - 3y^3 + 6z^4 > 2019$.
Solution $f(x) = x$ or $f(x) = c$ with $c = 2c^2 - 3c^3 + 6c^4$ so $c = 0$ or $c = \frac{1}{2}$

26. Find all function $f, g : (2019, \infty) \to \mathbb{R}$ such that

$$f(2x^2 - 3y^3 + 6z^4) = 2f(x)^2 + g(y) + 6zf(z)^3$$

Solution Just fix y and use the usual method, we will get $f(x) = x$ or $f(x) = 0$.
Then $g(y) = -3y^3$.

27. Find all function $f, g : (2019, \infty) \to \mathbb{R}$ such that

$$f(2x^2 - 3y^3 + 6z^4) = g(x) + h(y) + 6zf(z)^3$$

Solution
Fix $y = a > 2019$.Then we go back to our basis problem

$$f(2x^2 + 6z^4 - A) = g(x) + 6zf(z)^3 + B$$

where A, B are constant. So fix $x, z = b$ $2b^2 - A, 6b^4 - A > 2019, 4$, then

$$g(x) = f(2x^2 + 6b^4 - A) - 6bf(b)^3 - B$$

and

$$6zf(z)^3 = f(6z^4 + 2b^2 - A) - g(b) - B$$

Then now

$$f(2x^2 + 6z^4 - A) = f(2x^2 + C) + f(6z^4 + D) + E$$

and

$$f(6z^4 + D) = 6zf(z)^3 + F$$

This forces f is a linear function on x. Hence we can compute value of g, h.

28. Let P is a polynomial with real coefficients. Find all function $f, g : (2019, \infty) \to \mathbb{R}$ such that

$$f(2x^2 + P(y) + 6z^4) = 2f(x)^2 + g(y) + 6zf(z)^3$$

for all $x, y, z > 2019$ such that $2x^2 + 6z^4 + P(y) > 2019$.
Solution Just fix y as usual, then we have $f(x) = 0$ or $f(x) = x$. Therefore $g(y) = P(y)$.

29. Find all function $f, g : (2019, \infty) \to \mathbb{R}$ such that

$$f(2x^2 - \frac{y^3+2}{y} + 6z^4) = 2f(x)^2 + g(y) + 6zf(z)^3$$

for all $x, y, z > 2019$ such that $2x^2 + 6z^4 - \frac{y^3+2}{y} > 2019$.

30. Find all function $f : (2019, \infty) \to \mathbb{R}$ such that

$$f(x^2 - y^3 + z^4) = \frac{f(x)^4}{x^2+1} - \frac{f(y)^6}{y^3+1} + \frac{f(z)^8}{z^4+1}$$

31. Find all function $f : (2018, \infty) \to \mathbb{R}$ such that

$$f(x^2 + y^3 + z^4) = \frac{f(x)^4}{x^2+1} + \frac{f(y)^6}{y^3+1} + \frac{f(z)^8}{z^4+1}$$

32. Find all function $f : (2019, \infty) \to \mathbb{R}$ such that

$$f(x^3 + y^3 + z^3 + 3w^3) = \frac{f(x)^5}{x^2+1} + \frac{f(y)^5}{y^2+1} + \frac{f(z)^5}{z^2+1} + 3\frac{f(w)^5}{w^2+1}$$

33. Find all $a \in R$ such that there exists a function $f : (2019, \infty) \to \mathbb{R}^*$ such that

$$f(x^3 + y^3 + z^3)(\frac{x^2+a}{f(x)} + \frac{y^2+a}{f(y)} + \frac{z^2+a}{f(z)}) = 1$$

Solution Assume $a \in \mathbb{R}$ and $f : (2019, \infty) \to \mathbb{R}$ such that

$$f(x^3 + y^3 + z^3)(\frac{x^2+a}{f(x)} + \frac{y^2+a}{f(y)} + \frac{z^2+a}{f(z)}) = 1$$

then for all $x, y, z > 2019$.
Let $g(x) = \frac{1}{f(x)}$, then we have

$$g(x^3 + y^3 + z^3) = (x^2 + a)g(x) + (y^2 + a)g(y) + (z^2 + a)g(z)$$

This equation is familiar and we know that $g(x)$ is a linear function on x, hence $g(x) = cx + d$ with $c, d \in \mathbb{R}$.
Then $a = 0$ and hence $g(x) = cx$, so $f(x) = \frac{1}{cx}$ with $c \neq 0$.

34. Find all $a, b \in \mathbb{R}$ such that there exists a function $f : (2019, \infty) \to \mathbb{R}^*$ such that

$$f(x^3 + y^3 + z^3)(\frac{x^2 + ax + 1}{f(x)} + \frac{y^2 + ax + 1}{f(y)} + \frac{z^2 + ax + 1}{f(z)}) = 1$$

Solution Let $g(x) = \frac{1}{f(x)}$ then g is linear function on x hence $g(x) = \alpha x + \beta$. Plug in we have

$$\alpha(x^3 + y^3 + z^3) + \beta = \sum(x^2 + ax + b)(\alpha x + \beta)$$

Comparing the coefficients of x^2 forces $a = 0$, and comparing the coefficients of x forces $b = 0$.

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