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The equation $(w + x + y + z)(1/w + 1/x + 1/y + 1/z) = n$

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Bremner, Guy and Nowakowski [Which integers are representable as the product of the sum of three integers with the sum of their reciprocals? *Math. Comp.* **61**(203) (1993) 117–130] investigated the Diophantine problem of representing integers n in the form $(x + y + z)(1/x + 1/y + 1/z)$ for rationals x, y, z. For fixed n, the equation represents an elliptic curve, and the existence of solutions depends upon the rank of the curve being positive. They observed that the corresponding equation in four variables, the title equation here (representing a surface), has infinitely many solutions for each n, and remarked that it seemed plausible that there were always solutions with *positive* w, x, y, z when $n \geq 16$. This is false, and the situation is quite subtle. We show that there cannot exist such positive solutions when n is of the form $4m^2$, $4m^2 + 4$, where $m \not\equiv 2 \pmod 4$. Computations within our range seem to indicate that solutions exist for all other values of n.

Keywords: Elliptic curve; quartic surface; Diophantine representation; Hilbert symbol.

Mathematics Subject Classification 2010: 11D25, 11G05, 11D85, 14G05

1. Introduction

In Bremner, Guy and Nowakowski [1], the authors investigate the Diophantine problem of representing integers n in the form $(x+y+z)(1/x+1/y+1/z)$ for rationals x, y, z, which is equivalent to studying rational points on a parametrized family of elliptic curves. Solutions for x, y, z depend upon the rational rank of the curve being positive. In their concluding remarks, they observe that the corresponding equation in four variables, representing a surface,

$$(w + x + y + z)(1/w + 1/x + 1/y + 1/z) = n, \tag{1.1}$$

has infinitely many solutions for each n, following from the parametrization

$$(w, x, y, z) = (-(n - 1)t,\ t^2 + t + 1,\ (n - 1)t(t + 1),\ (t + 1)(n - 1)).$$

The minimum value of the form on the left-hand side of (1.1) as w, x, y, z take on positive real values is equal to 16. The question arises as to whether there are

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always *positive* solutions to (1.1) for w, x, y, z when $n \geq 16$. Quoting the previous authors: "… it seems likely that for $n \geq 16$ there is always such a representation, e.g., $16(1, 1, 1, 1)$; $17(2, 3, 3, 4)$; $18(1, 1, 2, 2)$; $19(8, 9, 18, 21)$; $20(1, 3, 3, 3)$."

For a fixed rational ratio y/z, Eq. (1.1) represents the equation of an affine cubic curve in w, x. For given n, it seems that an appropriate specialization of y/z to positive rational values may give rise to an elliptic curve of positive rank containing points with $x, w > 0$. Thus, the remark of Bremner, Guy and Nowakowski, seems plausible. However, the actual situation is rather more subtle. We performed a numerical search over the range $16 \leq n \leq 1000$; Table A.1 in the appendix lists results in the range $16 \leq n \leq 300$. It is striking that we find positive solutions to (1.1) for all values of n *except* $n = 36, 40, 64, 68, 100, 104, 196, 200, \ldots$. Allan MacLeod suggested that there might not be solutions when n was of the form $n = 4m^2, 4m^2 + 4$, for m odd or $m \equiv 0 \pmod 4$. We shall in fact prove that this is indeed the case: there do *not* exist positive solutions of (1.1) in the cases $n = 4m^2, 4m^2 + 4$, for $m \equiv 0, 1, 3 \pmod 4$. We thus obtain a family of (singular $K3$) surfaces where we show that the only rational points lie on lines typified by $w = x = 0$.

This investigation represents a "surface" analogue of the problem studied by Bremner and MacLeod [2], regarding positive representations of an integer n in the form $x/(y + z) + y/(z + x) + z/(x + y)$. It turned out that in this case, a positive representation is impossible when n is odd. This problem was discussed in an online forum, where Michael Stoll [5] gave an elegant alternative proof of this fact. Indeed, it was Stoll's proof that gave rise to the ideas behind the principal result of this paper.

The computations seem to indicate that when n is *not* of the form $n = 4m^2$, $4m^2 + 4$, with $m \not\equiv 2 \pmod 4$, then there is always a positive representation of n at (1.1). This seems very difficult to prove, however.

From perusal of solutions, we observe the following identity, involving Fibonacci numbers:

$$(w + x + y + z)(1/w + 1/x + 1/y + 1/z) = (4 + F_{2n}^2)^2,$$

where

$$(w, x, y, z) = (1, F_{2n-1}^2, F_{2n+1}^2, F_{2n-1}^2 F_{2n+1}^2).$$

In the case that $n = 3m$, in which case $8 \mid F_{6m}$, we set $k = \frac{1}{2}F_{6m}^2 + 2$ and obtain positive solutions to the family

$$(w + x + y + z)(1/w + 1/x + 1/y + 1/z) = 4k^2, \quad k \equiv 2 \pmod 4.$$

2. Inequalities

Suppose w, x, y, z is a solution in positive rationals of (1.1), and denote the symmetric functions of x, y, z by

$$s_1 = x + y + z, \quad s_2 = xy + xz + yz, \quad s_3 = xyz.$$

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Then (1.1) takes the form

$$s_2w^2-((n-1)s_3-s_1s_2)w+s_1s_3=0. \tag{2.1}$$

Taking the discriminant of (2.1) with respect to w:

$$((n+1)s_3-s_1s_2)^2-4ns_3^2=((\sqrt{n}+1)^2s_3-s_1s_2)((\sqrt{n}-1)^2s_3-s_1s_2)=\square. \tag{2.2}$$

Now, the coefficients of w^2, w^0 in Eq. (2.1) are positive, so that $w>0$ implies that the coefficient of w is negative, that is,

$$(n-1)s_3-s_1s_2>0. \tag{2.3}$$

This implies

$$(\sqrt{n}+1)^2s_3-s_1s_2>((\sqrt{n}+1)^2-(n-1))s_3=2(\sqrt{n}+1)s_3>0,$$

so that necessarily from (2.2)

$$(\sqrt{n}-1)^2s_3-s_1s_2>0.$$

In terms of x,y,z,

$$(n-2\sqrt{n})xyz-(x+y)(x+z)(y+z)>0. \tag{2.4}$$

This expression is quadratic in x with coefficients of x^2, x^0, equal to $-(y+z)$, $-yz(y+z)$, so we deduce that x positive demands the coefficient of x is positive, namely

$$-y^2+(n-2\sqrt{n}-2)yz-z^2>0. \tag{2.5}$$

Certainly, therefore,

$$-y^2+(n-2)yz-z^2>0. \tag{2.6}$$

Further, the x-discriminant at (2.4) must be positive, that is,

$$(-y^2+(n-2)yz-z^2)(-y^2+(n-4\sqrt{n}+2)yz-z^2)>0.$$

In virtue of (2.6), we must have

$$-y^2+(n-4\sqrt{n}+2)yz-z^2>0; \tag{2.7}$$

then

$$-y^2+(n-3\sqrt{n})yz-z^2>-y^2+(n-4\sqrt{n}+2)yz-z^2>0,$$

so that

$$\begin{aligned}y^4-2ny^3z+(n^2-8n-2)y^2z^2-2nyz^3+z^4\\ =(-y^2+(n-3\sqrt{n})yz-z^2)^2+6yz\sqrt{n}(-y^2+(n-3\sqrt{n})yz-z^2)\\ +(n-4)y^2z^2>0.\end{aligned} \tag{2.8}$$

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3. The Elliptic Curve

Equation (1.1) for $wxyz \neq 0$ is equivalent to

$$\begin{aligned} C_n : (y+z)(x+w)xw + yz(x^2+w^2) + (y^2-(n-4)yz+z^2)xw \\ + yz(y+z)(x+w) = 0, \end{aligned}$$

which may be considered as the equation of an affine cubic in w, x (say) over $\mathbb{Q}(y, z)$. A Weierstrass form is

$$F_n\colon Y^2 = X(X^2 + AX + B), \tag{3.1}$$

where

$$\begin{aligned} A &= y^4 - 2ny^3z + (n^2-8n-2)y^2z^2 - 2nyz^3 + z^4, \\ B &= 16ny^3z^3(y+z)^2. \end{aligned} \tag{3.2}$$

A mapping from F_n to C_n is given by the following (on homogenizing by writing x/d, w/d for x, w):

$$\begin{aligned} x : w : d = Y + X(-y^2+(n-2)yz-z^2)&: \\ -Y + X(-y^2+(n-2)yz-z^2) &: 2(y+z)(X-4ny^2z^2). \end{aligned} \tag{3.3}$$

We set

$$D = -y^2 + (n-2)yz - z^2,$$

where, from (2.6), $D > 0$ under specialization at positive y, z. The quadratic factor at (3.1)

$$X^2 + (y^4 - 2ny^3z + (n^2-8n-2)y^2z^2 - 2nyz^3 + z^4)X + 16ny^3z^3(y+z)^2$$

has discriminant $(-y^2+(n-2)yz-z^2)^2\Delta^2$, where

$$\begin{aligned} \Delta^2 &= y^4 - 2(n+2)y^3z + (n^2-12n+6)y^2z^2 - 2(n+2)yz^3 + z^4 \\ &= (-y^2 + (n+4\sqrt{n}+2)yz - z^2)(-y^2 + (n-4\sqrt{n}+2)yz - z^2) > 0, \end{aligned}$$

by (2.7). Thus, F_n has two components in the real plane. The 2-torsion on F_n occurs at $Y = 0$, when $X = 0$ or when $X = r_1, r_2$, where

$$r_1, r_2 = (-y^4 + 2ny^3z - (n^2-8n-2)y^2z^2 + 2nyz^3 - z^4 \pm D\Delta)/2,$$

which are both negative since their product is positive and their sum is

$$-(y^4 - 2ny^3z + (n^2-8n-2)y^2z^2 - 2nyz^3 + z^4) < 0,$$

by (2.8).

We seek points (X, Y) on F_n such that the corresponding x/d, w/d on C_n are positive; equivalently, that x/d and $x/d \cdot w/d$ are positive. This gives

$$(Y + X(-y^2+(n-2)yz-z^2))/(2(y+z)(X-4ny^2z^2)) > 0, \tag{3.4}$$

$$-X(X-4ny^2z^2)(X-4yz(y+z)^2) > 0. \tag{3.5}$$

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Now $-y^2 + (n-2)yz - z^2 > 0$, equivalent to $4ny^2z^2 > 4yz(y+z)^2$. Thus, (3.5) implies

$$X < 0, \quad \text{or } 4yz(y+z)^2 < X < 4ny^2z^2.$$

If the latter were to hold, then from (3.4),

$$\begin{aligned} Y + X(-y^2 + (n-2)yz - z^2) < 0 &\Rightarrow Y < -X(-y^2 + (n-2)yz - z^2) \\ &\Rightarrow Y^2 > X^2(-y^2 + (n-2)yz - z^2)^2 \\ &\Rightarrow X(X - 4ny^2z^2)(X - 4yz(y+z)^2) > 0, \end{aligned}$$

contradicting (3.5).

Necessarily, therefore, $X < 0$ (and $Y + X(-y^2 + (n-2)yz - z^2) < 0$). Positive values of $x : w$ thus correspond to rational points of F_n that lie on the bounded component.

We shall show that for $n = 4m^2$, $n = 4m^2 + 4$, where $m \not\equiv 2 \pmod 4$, then a rational point (X, Y) on F_n with $X < 0$ can occur only if $Y = 0$, that is, (X, Y) is a torsion point. This will be accomplished by proving that for a point (X, Y) with $Y \neq 0$, then the Hilbert symbol $(X, -D)_p = +1$ for all finite primes p whence $(X, -D)_\infty = +1$, implying, since $D > 0$, that $X > 0$.

Recall that for a, b non-zero elements of $\mathbb{Q}$, the Hilbert symbol $(a, b)_p$ for p a prime is defined to be $+1$ if the equation $ax^2 + by^2 = z^2$ has a solution with $x, y, z \in \mathbb{Q}_p$, and is defined to be -1 otherwise. We shall use the following facts about the Hilbert symbol; see, for example, Serre [4, Chap. III].

- The symbol is multiplicative, in that $(a, b)_p(a', b)_p = (aa', b)_p$.
- The symbol has a product formula, namely $\prod_p (a, b)_p = +1$.
- Suppose p is odd. Let $a = p^\alpha u$, $b = p^\beta v$, $p \nmid uv$. Then

$$(a, b)_p = (-1)^{\frac{p-1}{2}\alpha\beta} \left(\frac{u}{p}\right)^\beta \left(\frac{v}{p}\right)^\alpha.$$

- Suppose $p = 2$, with $a = 2^\alpha u$, $b = 2^\beta v$, uv odd. Then

$$(a, b)_2 = (-1)^{\left(\frac{u-1}{2}\right)\left(\frac{v-1}{2}\right) + \alpha\left(\frac{v^2-1}{8}\right) + \beta\left(\frac{u^2-1}{8}\right)}.$$

Remark 1. Observe that the equation

$$Bu^2 - Dv^2 = w^2$$

has the non-trivial solution $(u, v, w) = (1, 4yz(y+z),\ 4yz(y+z)^2)$, so that $(B, -D)_p = +1$ for all primes p.

4. The Case $n = 4m^2$

We use three lemmas, the first relating to odd primes p, the second and third relating to the prime 2.

Lemma 2. *Let* (X,Y) *with* $Y \neq 0$ *be a point on the curve* F_n *at* (3.1), *where* $n = 4m^2$. *Then the Hilbert symbol* $(X,-D)_p = +1$ *for all odd primes* p.

Proof. We suppose without loss of generality that $\gcd(y,z) = 1$. For $A(y,z)$, $B(y,z)$ are homogeneous of degrees 4, 8 in y, z, so if $y = dy_1$, $z = dz_1$, with $\gcd(y_1,z_1) = 1$, then setting $(X,Y) = (d^4X_1, d^6Y_1)$ we obtain a point (X_1,Y_1) on the curve

$$Y_1^2 = X_1(X_1^2 + A(y_1,z_1)X_1 + B(y_1,z_1)).$$

But $D(y,z) = d^2D(y_1,z_1)$, and

$$(X,-D(y,z))_p = (d^4X_1, -d^2D(y_1,z_1))_p = (X_1,-D(y_1,z_1))_p.$$

So it suffices to consider only $\gcd(y,z) = 1$.

If $X \notin \mathbb{Z}_p$, set $X = p^{-r}X_0$ with $p \nmid X_0$ and $r > 0$. Then

$$Y^2 = \frac{X_0(X_0^2 + p^rAX_0 + p^{2r}B)}{p^{3r}},$$

so that r is even and $X_0(X_0^2 + p^rAX_0 + p^{2r}B) = \square$. Thus, $X_0 \equiv \square \pmod{p}$, so $X_0 \in \mathbb{Z}_p^2$, and $(X,-D)_p = (p^rX_0,-D)_p = +1$. Thus, we need only consider the case where $X \in \mathbb{Z}_p$.

(1) Suppose $p \nmid X$, $p \nmid D$.
Now, X and D are units in $\mathbb{Z}_p$, so $(X,D)_p = +1$.

(2) Suppose $p \nmid X$, $p \mid D$.
Now, $X^2 + AX + B = (X + A/2)^2 - \Delta^2D^2/4 \equiv (X + A/2)^2 \pmod{p}$.

  (a) Suppose $p \mid A$. Then $p \nmid (X + A/2)$, and thus $X^2 + AX + B \in \mathbb{Z}_p^2$, with $(X,-D)_p = (X^2 + AX + B, -D)_p = +1$.

  (b) Suppose $p \nmid A$, $p \nmid (X + A/2)$. Then $(X,-D)_p = +1$.

  (c) Suppose $p \nmid A$, $p \mid (X + A/2)$. Then since $y^2 + z^2 \equiv (n-2)yz \pmod{p}$, we have

$$A = (y^2+z^2)^2 - 2nyz(y^2+z^2) + (n^2 - 8n - 4)y^2z^2 \equiv -8ny^2z^2 \pmod{p};$$

  thus $X \equiv -A/2 \equiv 4ny^2z^2 \equiv (4myz)^2 \pmod{p}$, and $p \nmid X$ implies $(X,-D)_p = +1$.

(3) Suppose $p \mid X$.
Suppose first that $p \nmid yz(y+z)$. Then the equation

$$(X^2 + AX + B)u^2 - Dv^2 = w^2$$

has the non-trivial solution $(1, 4yz(y+z), 4yz(y+z)^2)$ modulo p, so that $(X,-D)_p = (X^2 + AX + B, -D)_p = +1$.

Suppose second that $p \mid yz(y+z)$.

If $p \mid yz$ then $-D \equiv \square \not\equiv 0 \pmod{p}$, so that $(X,-D)_p = +1$.

If $p \nmid yz$, then $p \mid (y+z)$, and $-D \equiv -nyz \equiv 4m^2y^2 \pmod{p}$.

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(a) Suppose $p \nmid m$.
Then $-D \in \mathbb{Z}_p^2$ and $(X, -D)_p = +1$.

(b) Suppose $p \mid m$.
Let $r = \nu_p(m) > 0$, with $m = p^r m_1$, $p \nmid m_1$, and let $s = \nu_p(y+z) > 0$, with $y + z = p^s t$, $p \nmid t$.
If $r > s$, then $-D = p^{2s}(t^2 - 4p^{2r-2s} m_1^2 yz)$ and $p \nmid t$ implies $-D \in \mathbb{Z}_p^2$, so that $(X, -D)_p = +1$.
If $r < s$, then $-D = p^{2r}(p^{2s-2r}t^2 - 4m_1^2 yz) = p^{2r}(p^{2s-2r}t^2 - 4p^s m_1^2 yt + 4m_1^2 y^2)$, and since $p \nmid m_1 y$, then $-D \in Z_p^2$ and $(X, -D)_p = +1$.
If $r = s$, then $B = 16ny^3z^3(y+z)^2 = 64p^{4r}m_1^2(yz)^3t^2$, and $\nu_p(B) = 4r$. Further,

$$A = p^{2r} A_1, \quad A_1 \equiv 4y^2(t^2 - 4m_1^2 y^2) \pmod{p}, \tag{4.1}$$

so that in particular, $\nu_p(A) \geq 2r$.
Write $X = p^\alpha X_0$, $\alpha > 0$, $p \nmid X_0$; $A = p^\beta A_0$, $\beta \geq 2r$, $p \nmid A_0$; and $B = p^{4r} B_0$, $p \nmid B_0$. We have

$$Y^2 = p^\alpha X_0 (p^{2\alpha} X_0^2 + p^{\alpha+\beta} X_0 A_0 + p^{4r} B_0). \tag{4.2}$$

(i) Suppose $\alpha < 2r$.
Now $\nu_p(Y^2) = 3\alpha$, so that α is even, and $\alpha \leq 2r - 2$. Thus

$$\Box = X_0(X_0^2 + p^{\beta-\alpha} X_0 A_0 + p^{4r-2\alpha} B_0).$$

Because $\beta - \alpha \geq 2r - \alpha > 0$ and $4r - 2\alpha > 0$, then $X_0 \equiv \Box \pmod{p}$, so $X_0 \in \mathbb{Z}_p^2$, and $(X, -D)_p = +1$.

(ii) Suppose $\alpha = 2r$.
Now $\nu_p(X) = 2r$, and $-D = p^{2r}(t^2 - 4m_1^2 yz) = p^{2r} D_1$, say.

- Suppose $p \nmid D_1$. Then $\nu_p(-D) = 2r = \nu_p(X)$, so that $(X, -D)_p = +1$.
- Suppose $p \mid D_1$. Then $t^2 \equiv 4m_1^2 yz \equiv -4m_1^2 y^2 \pmod{p}$, and from (4.1), $A_1 \equiv -32m_1^2 y^4 \pmod{p}$. Since $p \nmid m_1 y$, it follows that $A_1 = A_0$ and $\beta = 2r$; and from (4.2),

$$\Box = X_0(X_0^2 + X_0 A_0 + B_0).$$

Since $p \mid D_1$, we have $p \mid (A_0^2 - 4B_0)$, since this latter term equals $D_1^2 \Delta^2$. Thus, $X_0^2 + A_0 X_0 + B_0 \equiv (X_0 + A_0/2)^2 \pmod{p}$.
In the case $p \nmid (X_0 + A_0/2)$, then $X_0^2 + A_0 X_0 + B_0 \in \mathbb{Z}_p^2$, so $X_0 \in Z_p^2$, so $(X, -D)_p = +1$.
In the case $p \mid (X_0 + A_0/2)$, then $A_0 \equiv -32m_1^2 y^4 \pmod{p}$ implies $X_0 \equiv 16m_1^2 y^4 \pmod{p}$. Thus, $X_0 \in \mathbb{Z}_p^2$, and $(X, -D) = (p^{2r} X_0, -D)_p = +1$.

(iii) Suppose $\alpha > 2r$.
Now, $\nu_p(Y^2) = 4r + \alpha$, so that α is even, with $\alpha \geq 2r + 2$. From (4.2),

$$\Box = X_0(p^{2\alpha-4r} X_0^2 + p^{\alpha+\beta-4r} A_0 X_0 + B_0).$$

However, $2\alpha-4r>0$ and $\alpha+\beta-4r>0$, we have $X_0B_0 \equiv \square \pmod{p}$. Thus, $X_0B_0 \in \mathbb{Z}_p^2$ and $(XB,-D)_p=(p^{\alpha+4r}X_0B_0,-D)_p=+1$. Since $(B,-D)_p=+1$, it follows that $(X,-D)_p=+1$. $\square$

For the prime 2, consider separately the instances $m \equiv 0 \pmod{4}$, and m odd. Suppose first that $m \equiv 0 \pmod{4}$.

Now if (w,x,y,z) is an integer solution of (1.1), then certainly we may assume their greatest common divisor equals 1. Parity considerations show that either (i) one of w,x,y,z is odd, the others even; or (ii) two of w,x,y,z are odd, the others even; or (iii) w,x,y,z are all odd. We order w,x,y,z so that in case (i) then $(w,x,y,z)\equiv(0,0,0,1) \pmod{2}$; and in case (ii) then $(w,x,y,z)\equiv(1,0,0,1) \pmod{2}$. In case (iii), order w,x,y,z so that $4 \nmid (y+z)$: this is possible, otherwise all six sums of any two of w,x,y,z are divisible by 4, implying w,x,y,z are each even. It follows that in each case, the ordering satisfies $4\nmid(y+z)$.

Lemma 3. *Let* (X,Y), $Y\neq 0$, *be a point on the curve* F_n *at* (3.1), *where* $n=4m^2$, $m\equiv 0 \pmod{4}$, $4\nmid(y+z)$. *Then the Hilbert symbol* $(X,-D)_2=+1$.

Proof. Since $-D=(y+z)^2-4m^2yz$, then $y+z$ odd implies $-D\equiv 1 \pmod{8}$, and $y+z\equiv 2 \pmod{4}$ implies $-D\equiv 4 \pmod{32}$. In either case, $-D\in\mathbb{Z}_2^2$, and $(X,-D)_2=+1$. $\square$

Remark 4. The Lemma holds of course for X any non-zero rational.

Suppose second that m is odd. In this instance, we suppose without loss of generality that $4\nmid(y-z)$. This is possible, for otherwise all six differences of type $y-z$ are divisible by 4, implying $w\equiv x\equiv y\equiv z\equiv \pm 1 \pmod{4}$, and (1.1) gives an impossible congruence mod 16.

Lemma 5. *Let* (X,Y), $Y\neq 0$, *be a point on the curve* F_n *at* (3.1), *where* $n=4m^2$, *m odd*, $4\nmid(y-z)$. *Then the Hilbert symbol* $(X,-D)_2=+1$.

Proof. We have $-D=(y-z)^2-4(m^2-1)yz$. Thus, $y-z$ odd implies $-D\equiv 1 \pmod{8}$, and $y-z\equiv 2 \pmod{4}$ implies $-D\equiv 4 \pmod{32}$. In either case, $-D\in\mathbb{Z}_2^2$, and $(X,-D)_2=+1$. $\square$

Theorem 6. *Suppose* $n=4m^2$, $m\not\equiv 2 \pmod{4}$. *In the case* $m\equiv 0 \pmod{4}$, *we suppose* $4\nmid(y+z)$; *and in the case* $m\equiv 1 \pmod{2}$, *we suppose* $4\nmid(y-z)$. *Then if* (X,Y) *is a point on the curve* F_n *at* (3.1) *with* X *negative, we have* $Y=0$.

Proof. Suppose $(X,Y)\in F_n(\mathbb{Q})$, $Y\neq 0$. In the case $m\equiv 0 \pmod{4}$, then by Lemmas 2 and 3, the Hilbert symbol $(X,-D)_p=+1$ for all finite primes p. It follows that $(X,-D)_\infty=+1$. But $D>0$, and thus necessarily $X>0$. Similarly in the case $m\equiv 1 \pmod{2}$, Lemmas 2 and 5 imply the Hilbert symbol $(X,-D)_p=+1$

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for all finite primes p. It follows that $(X,-D)_\infty=+1$. But $D>0$, and so necessarily $X>0$. ☐

Corollary 7. *Let* $n=4m^2$, $m\not\equiv 2 \pmod 4$. *Suppose* w,x,y,z *is a solution of* (1.1), *where we suppose* $4\nmid(y+z)$ *in the case* $m\equiv 0 \pmod 4$, *and where we suppose* $4\nmid(y-z)$ *in the case* $m\equiv 1 \pmod 2$. *Then* $w=x$.

Proof. Immediate from Theorem 6 and the mappings (3.3). ☐

In the situation of $n=4m^2$, $m\equiv 0 \pmod 4$, then as remarked above, it may be assumed that an integer solution (w,x,y,z) to (1.1) satisfies $4\nmid(y+z)$. There must be another pair of the integers whose sum is not divisible by 4, otherwise $y\equiv -w\equiv x\equiv -z \pmod 4$, contradiction. So without loss of generality, we can assume not only $4\nmid(y+z)$, but also either $4\nmid(w+x)$ or $4\nmid(w+y)$. Applying Corollary 7 now gives that a solution (w,x,y,z) to (1.1) in positive rationals must satisfy either $w=x, y=z$, or $w=x=z$.

Theorem 8. *Let* $n=4m^2$, $m\equiv 0 \pmod 4$. *Then* (1.1) *has no solutions in positive rationals.*

Proof. Such a solution satisfies either $w=x, y=z$ or $w=x=z$. In the former case, (1.1) becomes $x/z+z/x=m^2-2$, implying $x=z$, $m=2$, impossible. In the latter case, (1.1) becomes $3(y/z+z/y)=4m^2-10$, which implies $m=2,\sqrt{5}$, impossible. ☐

In this situation of $n=4m^2$, m odd, we have assumed an ordering on w,x,y,z such that $4\nmid(y-z)$. But there must be another pair whose difference is not divisible by 4, otherwise again $w\equiv x\equiv y\equiv z\equiv \pm 1 \pmod 4$ leading to a contradiction at (1.1). So without loss of generality, we assume also either $4\nmid(w-x)$ or $4\nmid(w-y)$. Applying Corollary 7 now gives that a solution (w,x,y,z) to (1.1) in positive rationals must satisfy either $w=x, y=z$, or $w=x=z$.

Theorem 9. *Let* $n=4m^2$, m *odd. Then* (1.1) *has no solutions in positive rationals.*

Proof. As above. ☐

5. The Case $n=4m^2+4$

We use three lemmas, the first relating to odd primes p, the second and third relating to the prime 2.

Lemma 10. *Let* (X,Y) *with* $Y\neq 0$ *be a point on the curve* F_n *at* (3.1), *where* $n=4m^2+4$. *Then the Hilbert symbol* $(X,-D)_p=+1$ *for all odd primes* p.

Proof. As before, we suppose that $(y,z)=1$; and as in Lemma 2, it is only necessary to consider $X\in\mathbb{Z}_p$.

(1) Suppose $p \nmid X$, $p \nmid D$.
As before, $(X, -D)_p = +1$.

(2) Suppose $p \nmid X$, $p \mid D$.
Then $X^2 + AX + B \equiv (X + A/2)^2 \pmod{p}$, and so $p \nmid (X + A/2)$ implies $(X, -D)_p = (X^2 + AX + B, -D)_p = +1$. And if $p \mid (X + A/2)$, then certainly $p \nmid A$; and from $y^2 + z^2 \equiv (n-2)yz \pmod{p}$, we obtain $A \equiv -8ny^2z^2 \pmod{p}$, so that $p \nmid nyz$, and $X \equiv 4ny^2z^2 \pmod{p}$.

  (a) Suppose $p \nmid m$.
  Then $-D = (y-z)^2 - 4m^2yz \equiv 0 \pmod{p}$, so $p \nmid (y-z)$ and $yz \equiv \square \pmod{p}$. Also, $-D = (y+z)^2 - nyz \equiv 0 \pmod{p}$, so that $nyz \equiv \square \pmod{p}$, whence $n \equiv \square \pmod{p}$. Thus $X \in \mathbb{Z}_p^2$, and $(X, -D)_p = +1$.

  (b) Suppose $p \mid m$.
  Then $n = 4m^2 + 4 \equiv 4 \pmod{p}$, and $X \equiv 4ny^2z^2 \equiv (4yz)^2 \pmod{p}$. Since $p \nmid yz$, then $X \in Z_p^2$, and $(X, -D)_p = +1$.

(3) Suppose $p \mid X$.
Suppose first that $p \nmid yz(y+z)$. The equation $(X^2+AX+B)u^2 - Dv^2 = w^2$ has the non-trivial solution $(1,\ 4yz(y+z),\ 4yz(y+z)^2)$ modulo p, and thus has a non-trivial solution in $\mathbb{Q}_p$. Therefore, $(X, -D)_p = (X^2 + AX + B, -D)_p = +1$.
Suppose second that $p \mid yz(y+z)$.
If $p \mid yz$, then $-D = y^2 + z^2 - (n-2)yz \equiv \square \not\equiv 0 \pmod{p}$, and $-D \in \mathbb{Z}_p^2$, with $(X, -D)_p = +1$.
If $p \nmid yz$, then $p \mid (y+z)$, and $-D = (y+z)^2 - nyz \equiv ny^2 \pmod{p}$.

  (a) Suppose $p \nmid D$.
  Then $p \nmid ny$. Further,

$$\begin{aligned} A &= y^4 + z^4 - 2yz(y^2+z^2) + (n^2 - 8n - 2)y^2z^2 \\ &\equiv n(n-4)y^4 \equiv n(4m^2y^4) \pmod{p}. \end{aligned} \tag{5.1}$$

  If $p \mid m$, then $n = 4m^2 + 4 \equiv 4 \pmod{p}$, $-D \equiv 4y^2 \pmod{p}$, $-D \in \mathbb{Z}_p^2$, and $(X, -D)_p = +1$.
  If $p \nmid m$, then from (5.1), we have $p \nmid A$.
  Write $y + z = p^\alpha t$, $p \nmid t$, $\alpha \geq 1$, and let $X = p^s X_0$, $p \nmid X_0$, $s \geq 1$. Then

$$B = 16ny^3z^3(y+z)^2 = p^{2\alpha}B_0, \quad p \nmid B_0,$$

  and

$$Y^2 = p^s X_0 (p^{2s}X_0^2 + p^s A X_0 + p^{2\alpha} B_0). \tag{5.2}$$

- Suppose $s < 2\alpha$. Then $\nu_p(Y^2) = 2s$ and $X_0(p^s X_0^2 + AX_0 + p^{2\alpha - s}B_0) = \square$. Modulo p, $AX_0^2 \equiv \square$, so $A \equiv \square$, and from (5.1), $n \equiv \square$. Therefore, $-D \equiv ny^2 \equiv \square$, so $-D \in \mathbb{Z}_p^2$ and $(X, -D)_p = +1$.
- Suppose $s = 2\alpha$. Then $\nu_p(X)$ and $\nu_p(-D)$ are even (the latter, zero), giving $(X, -D)_p = +1$.

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- Suppose $s > 2\alpha$. Then $\nu_p(Y^2) = 2\alpha + s$, so s is even. Thus, $\nu_p(X)$ and $\nu_p(-D)$ are even (the latter, zero), giving $(X, -D)_p = +1$.

(b) Suppose $p \mid D$.
Since $p \nmid yz$ and $-D = (y + z)^2 - nyz$, we have $p \mid n$.
Write $y + z = p^u s$, $p \nmid s$, $u \geq 1$, and let $n = 4p^v t$, $p \nmid t$, $v \geq 1$. Then

$$-D = p^{2u}s^2 - 4p^v tyz,$$

and it is only necessary to consider the case $v \leq 2u$, otherwise $-D \in Z_p^2$ and $(X, -D)_p = +1$.

(i) Suppose $v < 2u$.
Here,

$$-D = p^v(p^{2u-v}s^2 - 4tyz) = p^v D_0, \quad \text{where } D_0 \equiv 4ty^2 \pmod{p}. \tag{5.3}$$

Further,

$$\begin{aligned} A &= (y - z)^2(y + z)^2 - 2nyz(y + z)^2 + n(n - 4)y^2z^2 \\ &= (y - z)^2p^{2u}s^2 - 8p^{2u+v}s^2tyz + 16p^v t(p^v t - 1)y^2z^2 \\ &= p^v((y - z)^2p^{2u-v}s^2 - 8p^{2u}s^2tyz + 16t(p^v t - 1)y^2z^2) \\ &= p^v A_0, \quad \text{with } A_0 \equiv -16ty^4 \pmod{p}. \end{aligned} \tag{5.4}$$

Also,

$$B = 16ny^3z^3(y + z)^2 = 64p^{2u+v}s^2ty^3z^3 = p^{2u+v}B_0, \quad p \nmid B_0. \tag{5.5}$$

Write $X = p^\alpha X_0$, $p \nmid X_0$, $\alpha \geq 1$. Then

$$Y^2 = p^\alpha X_0(p^{2\alpha}X_0^2 + p^{\alpha+v}A_0X_0 + p^{2u+v}B_0).$$

- $v > \alpha$.
  We have $\nu_p(Y^2) = 3\alpha$ (recall $2u > v$, so $2u + v > 2\alpha$), so α is even, and

$$X_0(X_0^2 + p^{v-\alpha}A_0X_0 + p^{2u+v-2\alpha}B_0) = \Box.$$

  This implies $X_0 \in \mathbb{Z}_p^2$ and $(X, -D)_p = (X_0, -D)_p = +1$.

- $v = \alpha$.
  We have

$$Y^2 = p^{3\alpha}X_0(X_0^2 + A_0X_0 + p^{2u-\alpha}B_0). \tag{5.6}$$

  If α is even, then $(X, -D)_p = (X_0, D_0)_p = +1$. And if α is odd, then (5.6) implies $p \mid (X_0^2 + A_0X_0)$, so that $p \mid (X_0 + A_0)$. Using (5.3) and (5.4), this gives $X_0D_0 \equiv -A_0D_0 \equiv 64t^2y^6 \pmod{p}$. Therefore,

$$\begin{aligned} (X, -D)_p = (-1)^{\alpha v\frac{p-1}{2}}\left(\frac{X_0}{p}\right)^v\left(\frac{D_0}{p}\right)^\alpha &= (-1)^{\frac{p-1}{2}}\left(\frac{X_0D_0}{p}\right)^\alpha \\ &= (-1)^{\frac{p-1}{2}}. \end{aligned}$$

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However, p is an odd divisor of $n = 4(m^2 + 1)$, and thus $p \equiv 1 \pmod 4$, and $(X, -D)_p = +1$.

- $v < \alpha < 2u$.
  Here, $2\alpha, 2u + v > \alpha + v$, so that $\nu_p(Y^2) = 2\alpha + v$, and v is even, with
  $$X_0(p^{\alpha - v}X_0^2 + A_0X_0 + p^{2u-\alpha}B_0) = \square.$$
  Thus $A_0 \equiv \square \pmod p$, and from (5.4), $-t \equiv \square \pmod p$. But from (5.3), $D_0 \equiv 4ty^2 \equiv -4tm^2y^2 \equiv \square \pmod p$. So $D_0 \in \mathbb{Z}_p^2$, $-D = p^vD_0 \in \mathbb{Z}_p^2$, and $(X, -D)_p = +1$.
- $v < \alpha = 2u$.
  Now,
  $$Y^2 = p^{4u+v}X_0(p^{2u-v}X_0^2 + A_0X_0 + B_0).$$
  If $p \nmid (A_0X_0 + B_0)$, then $\nu_p(Y^2) = 4u + v$, and v is even. Then $\nu_p(X) \equiv \nu_p(D) \equiv 0 \pmod 2$, with $(X, -D)_p = +1$.
  If $p \,|\, (A_0X_0 + B_0)$, then using (5.4), (5.5), we have $-16ty^4X_0 + 64ty^3z^3s^2 \equiv 0 \pmod p$, so that $X_0 \equiv -4s^2y^2 \equiv 4m^2s^2y^2 \pmod p$. Thus $X_0 \in \mathbb{Z}_p^2$ and $(X, -D)_p = (p^{2u}X_0, -D)_p = +1$.
- $v < \alpha > 2u$.
  We have $2\alpha > \alpha + v > 2u + v$, and
  $$Y^2 = p^{\alpha+2u+v}X_0(p^{2\alpha-2u-v}X_0^2 + p^{\alpha-2u}A_0X_0 + B_0).$$
  Hence $\nu_p(Y^2) = \alpha + 2u + v$, and $\alpha + v$ is even. If both are even, then $(X, -D)_p = +1$; so suppose both are odd. Thus $X_0B_0 \equiv \square \pmod p$, so $(X_0B_0, -D)_p = +1$. However, $XB = p^{\alpha+2u+v}X_0B_0$, with $\alpha + v$ even; so $(BX, -D)_p = +1$. From $(B, -D)_p = +1$ follows $(X, -D)_p = +1$.

(ii) Suppose $v = 2u$.
Here,
$$-D = p^{2u}(s^2 - 4tyz) = p^{2u}D_0, \quad D_0 = s^2 - 4tyz. \tag{5.7}$$
Further,
$$\begin{aligned} A &= (y-z)^2(y+z)^2 - 2nyz(y+z)^2 + n(n-4)y^2z^2 \\ &= (y-z)^2p^{2u}s^2 - 8p^{4u}s^2tyz + 16p^{2u}m^2ty^2z^2 \\ &= p^{2u}((y-z)^2s^2 - 8p^{2u}s^2tyz + 16m^2ty^2z^2) \\ &= p^{2u}A_0, \quad \text{with } A_0 \equiv (y-z)^2s^2 - 16ty^2z^2 \equiv 4y^2(s^2 - 4ty^2) \\ &\qquad \pmod p. \end{aligned} \tag{5.8}$$
Also,
$$B = 16ny^3z^3(y+z)^2 = p^{4u}B_0, \quad B_0 = 64s^2ty^3z^3, \quad p \nmid B_0. \tag{5.9}$$

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Write $X = p^\alpha X_0$, $p \nmid X_0$, $\alpha \geq 1$. Then

$$Y^2 = p^\alpha X_0(p^{2\alpha} X_0^2 + p^{2u+\alpha} A_0 X_0 + p^{4u} B_0). \tag{5.10}$$

- $\alpha < 2u$.
  As before, $\nu_p(X)$ is even, and $X \in \mathbb{Z}_p^2$ with $(X, -D)_p = +1$.
- $\alpha = 2u$.
  Now, $X_0(X_0^2 + A_0X_0 + B_0) = \square$. If $p \nmid D_0 = s^2 - 4tyz$ then $\nu_p(D) = 2u = \nu_p(X)$ and $(X, -D)_p = +1$. It suffices to consider $p \mid D_0$, that is $p \mid s^2 - 4tyz$.
  Then $p^{4u}(A_0^2 - 4B_0) = A^2 - 4B = D^2\Delta^2 = p^{4u}D_0^2\Delta^2$, so that $A_0^2 - 4B_0 \equiv 0 \pmod p$. Thus, $X_0^2 + A_0X_0 + B_0 \equiv (X_0 + A_0/2)^2 \pmod p$. If $p \nmid (X_0 + A_0/2)$ then $X_0^2 + A_0X_0 + B_0 \in \mathbb{Z}_p^2$, so $X_0 \in \mathbb{Z}_p^2$, so $(X, -D)_p = (p^{2u}X_0, -D)_p = +1$.
  If $p \mid (X_0 + A_0/2)$ then $X_0 \equiv -A_0/2$ and from (5.8),
  $$X_0 \equiv -2y^2(s^2 - 4ty^2) \equiv 16ty^4 \equiv -4s^2y^2 \equiv 4m^2s^2y^2 \pmod p.$$
  Hence $X_0 \in \mathbb{Z}_p^2$, and $(X, -D)_p = (p^{2u}X_0, -D)_p = +1$.
- $\alpha > 2u$.
  Now $2\alpha > \alpha + 2u > 4u$. So $\nu_p(Y^2) = \alpha + 4u$, α is even, and
  $$X_0(p^{2\alpha-4u}X_0^2 + p^{\alpha-2u}A_0X_0 + B_0) = \square.$$
  Thus, $X_0B_0 \equiv \square \pmod p$, so $XB = p^{\alpha+4u}X_0B_0 \in \mathbb{Z}_p^2$. Then $(X, -D)_p = (XB, -D)_p(B, -D)_p = +1$. □

Again, when dealing with the case of $p = 2$, we split into two cases, $n = 4m^2 + 4$ with $m \equiv 0 \pmod 4$, and $m \equiv 1 \pmod 2$. When $m \equiv 0 \pmod 4$, we argue as before to suppose that $4 \nmid (y - z)$. When $m \equiv 1 \pmod 2$, we suppose that $4 \nmid (y + z)$.

Lemma 11. *Let* (X, Y), $Y \neq 0$, *be a point on the curve* F_n *at* (3.1), *where* $n = 4m^2 + 4$, $m \equiv 0 \pmod 4$, $4 \nmid (y - z)$. *Then the Hilbert symbol* $(X, -D)_2 = +1$.

Proof. We have $-D = (y - z)^2 - 4m^2yz$, so that $-D \equiv 1 \pmod 8$, $-D \equiv 4 \pmod{32}$ according as $y - z$ odd, $y - z \equiv 2 \pmod 4$. Thus, $-D \in \mathbb{Z}_2^2$, and $(X, -D)_2 = +1$. □

Lemma 12. *Let* (X, Y), $Y \neq 0$, *be a point on the curve* F_n *at* (3.1), *where* $n = 4m^2 + 4$, m *odd*, $4 \nmid (y + z)$. *Then the Hilbert symbol* $(X, -D)_2 = +1$.

Proof. We have $-D = (y + z)^2 - 4(m^2 + 1)yz$. So $y + z$ odd gives $-D \in \mathbb{Z}_2^2$, $(X, -D)_2 = +1$. It remains to consider $y + z \equiv 2 \pmod 4$. Then y, z are both odd, and $y \equiv z \pmod 4$. Thus, $D_0 = -D/4 \equiv -1 \pmod 8$.
Write $m^2 + 1 = 2m_1$, m_1 odd, $y + z = 2t$, t odd. Then

$$B = 64(m^2 + 1)y^3z^3(y + z)^2 = 2^9B_0, \quad B_0 = m_1t^2y^3z^3, \tag{5.11}$$

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and, with $y - z = 4s$,

$$\begin{aligned} A &= (y^2 - z^2)^2 - 2nyz(y+z)^2 + n(n-4)y^2z^2 \\ &= 64s^2t^2 - 32(m^2+1)t^2yz + 32m_1m^2y^2z^2 \\ &= 2^5(2s^2t^2 - 2m_1t^2yz + m_1m^2y^2z^2) \\ &= 2^5A_0, \quad A_0 = 2s^2t^2 - 2m_1t^2yz + m_1m^2y^2z^2. \end{aligned} \tag{5.12}$$

Set $X = 2^kX_0$, X_0 odd, so that $Y^2 = 2^kX_0(2^{2k}X_0^2 + 2^{5+k}A_0X_0 + 2^9B_0)$.

- $k \leq 3$.
  Here, $Y^2 = 2^{3k}X_0(X_0^2 + 2^{5-k}A_0X_0 + 2^{9-2k}B_0)$, so that $\nu_2(Y^2) = 3k$, k is even, $k \leq 2$, and
  $$X_0(X_0^2 + 2^{5-k}A_0X_0 + 2^{9-2k}B_0) = \square.$$
  Thus, $X_0 \equiv \square \equiv 1 \pmod 8$, $X_0 \in \mathbb{Z}_2^2$, and $(X, -D)_2 = (X_0, -D)_2 = +1$.
- $k = 4$.
  Here, $X_0(X_0^2 + 2A_0X_0 + 2B_0) = \square$. Thus, $X_0 \equiv X_0^3 \equiv 1 \pmod 4$. Accordingly, $X = 2^4X_0$, $X_0 \equiv 1 \pmod 4$, and $-D = 2^2D_0$ with $D_0 \equiv -1 \pmod 8$. Hence $(X, -D)_2 = (X_0, D_0)_2 = (-1)^{(\frac{X_0-1}{2})(\frac{D_0-1}{2})} = +1$.
- $k = 5$.
  Here, $X_0(2X_0^2 + 2A_0X_0 + B_0) = \square$, so that $1 \equiv 2(X_0 + A_0) + X_0B_0 \equiv X_0B_0 \pmod 4$, and $X_0 \equiv B_0 \equiv m_1t^2y^3z^3 \equiv m_1t^2y^6 \equiv m_1 \equiv 1$, using $y \equiv z \pmod 4$ and $m^2 + 1 = 2m_1$. But now $(X, -D)_2 = (2X_0, D_0)_2 = (-1)^{(\frac{X_0-1}{2})(\frac{D_0-1}{2}) + (\frac{D_0^2-1}{8})} = +1$.
- $k \geq 6$.
  Here, $\nu_2(Y^2) = k + 9$, so k is odd, $k \geq 7$, with $X_0(2^{2k-9}X_0^2 + 2^{k-4}A_0X_0 + B_0) = \square$. Thus, $X_0B_0 \equiv 1 \pmod 8$, $X_0B_0 \in \mathbb{Z}_2^2$, $XB = 2^{k+9}X_0B_0 \in \mathbb{Z}_2^2$, and $(X, -D)_2 = (XB, -D)_2(B, -D)_2 = +1$. $\square$

Theorem 13. *Suppose* $n = 4m^2 + 4$, $m \not\equiv 2 \pmod 4$. *In the case*, $m \equiv 0 \pmod 4$ *we suppose* $4 \nmid (y + z)$; *and in the case* $m \equiv 1 \pmod 2$, *we suppose* $4 \nmid (y - z)$. *Then if* (X, Y) *is a point on the curve* F_n *at* (3.1) *with* X *negative, we have* $Y = 0$.

Proof. Suppose $(X, Y) \in F_n(\mathbb{Q})$, $Y \neq 0$. In the case, $m \equiv 0 \pmod 4$, then by Lemmas 10, 11, the Hilbert symbol $(X, -D)_p = +1$ for all finite primes p. It follows that $(X, -D)_\infty = +1$. But $D > 0$, and thus necessarily $X > 0$. Similarly in the case $m \equiv 1 \pmod 2$, Lemmas 10 and 12 imply the Hilbert symbol $(X, -D)_p = +1$ for all finite primes p. It follows that $(X, -D)_\infty = +1$. But $D > 0$, and thus necessarily $X > 0$. $\square$

Arguing as in the previous section, the following theorem is now immediate.

Theorem 14. *Suppose* $n = 4m^2 + 4$, $m \not\equiv 2 \pmod 4$. *Then* (1.1) *has no solutions in positive rationals.*

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Appendix A.

We have computed solutions to Eq. (1.1) in the range $16 \leq n \leq 1000$. An initial brute force search was conducted for solutions satisfying $w+x+y+z \leq 2000$. Where no solution was found, the computation was then augmented by a search over the parameter $t = y/z$, when (1) becomes an elliptic curve. Routines in Magma [3] were used to compute rational points with positive coordinates. We did not attempt to find the "smallest" solution for a given n. Some solutions were non-trivial to find, for example at $n = 785$, with $(w, x, y, z) = (105, 2788, 22176, 53856)$. Table A.1 lists solutions in the range $16 \leq n \leq 300$.

Table A.1. Solutions of $(w + x + y + z)(1/w + 1/x + 1/y + 1/z) = n$.

n	(w, x, y, z)	n	(w, x, y, z)	n	(w, x, y, z)
16	(1, 1, 1, 1)	17	(2, 3, 3, 4)	18	(1, 1, 2, 2)
19	(5, 8, 12, 15)	20	(1, 1, 1, 3)	21	(8, 14, 15, 35)
22	(1, 1, 2, 4)	23	(76, 220, 285, 385)	24	(1, 2, 3, 6)
25	(1, 1, 4, 4)	26	(20, 27, 39, 130)	27	(3, 7, 8, 24)
28	(2, 9, 10, 15)	29	(1, 1, 4, 6)	30	(2, 3, 10, 15)
31	(1, 4, 5, 10)	32	(1, 2, 6, 9)	33	(12, 35, 51, 140)
34	(6, 35, 40, 63)	35	(8, 45, 63, 84)	36	$*$
37	(1, 3, 8, 12)	38	(2, 3, 15, 20)	39	(4, 18, 20, 63)
40	$*$	41	(1, 5, 12, 12)	42	(1, 1, 4, 12)
43	(5, 14, 44, 77)	44	(2, 14, 15, 35)	45	(1, 1, 6, 12)
46	(6, 35, 78, 91)	47	(6, 28, 51, 119)	48	(1, 1, 3, 15)
49	(1, 2, 5, 20)	50	(1, 2, 9, 18)	51	(35, 77, 480, 528)
52	(1, 3, 4, 24)	53	(2, 4, 9, 45)	54	(1, 3, 8, 24)
55	(9, 44, 77, 234)	56	(6, 78, 91, 105)	57	(3, 6, 40, 56)
58	(2, 11, 20, 55)	59	(6, 65, 104, 120)	60	(3, 5, 6, 70)
61	(2, 7, 15, 60)	62	(3, 16, 45, 80)	63	(3, 12, 50, 75)
64	$*$	65	(2, 9, 44, 44)	66	(2, 2, 5, 45)
67	(1, 4, 20, 25)	68	$*$	69	(24, 140, 561, 595)
70	(1, 6, 21, 28)	71	(1, 10, 21, 28)	72	(1, 4, 21, 28)
73	(5, 44, 45, 198)	74	(28, 33, 209, 756)	75	(4, 7, 78, 91)
76	(1, 7, 10, 42)	77	(1, 5, 18, 36)	78	(1, 6, 28, 28)
79	(1, 3, 24, 28)	80	(1, 5, 9, 45)	81	(3, 6, 20, 116)
82	(7, 24, 112, 273)	83	(8, 78, 129, 344)	84	(1, 3, 5, 45)
85	(1, 18, 20, 36)	86	(5, 28, 30, 252)	87	(2, 4, 15, 84)
88	(2, 9, 22, 99)	89	(1, 1, 12, 28)	90	(3, 21, 80, 120)
91	(20, 21, 261, 580)	92	(1, 3, 12, 48)	93	(3, 7, 30, 140)
94	(1, 5, 8, 56)	95	(3, 8, 88, 99)	96	(1, 7, 30, 42)
97	(5, 20, 21, 276)	98	(1, 18, 33, 36)	99	(1, 4, 20, 50)
100	$*$	101	(7, 15, 220, 220)	102	(5, 9, 16, 240)
103	(5, 92, 110, 253)	104	$*$	105	(2, 44, 44, 99)
106	(1, 9, 20, 60)	107	(2, 11, 20, 132)	108	(3, 40, 105, 140)
109	(5, 12, 63, 280)	110	(14, 168, 248, 903)	111	(45, 60, 385, 2156)
112	(1, 14, 35, 50)	113	(1, 3, 16, 60)	114	(7, 102, 231, 374)
115	(2, 9, 52, 117)	116	(2, 9, 39, 130)	117	(1, 3, 24, 56)
118	(1, 1, 12, 42)	119	(2, 4, 63, 84)	120	(1, 2, 12, 60)
121	(1, 21, 28, 60)	122	(1, 12, 13, 78)	123	(3, 36, 136, 153)
124	(9, 154, 273, 572)	125	(2, 9, 13, 156)	126	(1, 2, 10, 65)
127	(1, 8, 27, 72)	128	(3, 11, 35, 231)	129	(1, 9, 14, 84)

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Table A.1. (*Continued*)

n	(w, x, y, z)	n	(w, x, y, z)	n	(w, x, y, z)
130	(1, 5, 28, 70)	131	(7, 15, 60, 492)	132	(1, 2, 18, 63)
133	(5, 96, 195, 312)	134	(5, 8, 65, 312)	135	(5, 68, 102, 420)
136	(3, 11, 110, 186)	137	(2, 13, 57, 156)	138	(3, 7, 90, 180)
139	(3, 17, 80, 240)	140	(7, 160, 189, 540)	141	(3, 8, 88, 198)
142	(3, 7, 24, 238)	143	(1, 3, 40, 60)	144	(1, 21, 33, 77)
145	(3, 10, 156, 156)	146	(4, 5, 126, 180)	147	(1, 14, 33, 84)
148	(5, 7, 13, 325)	149	(4, 15, 152, 285)	150	(1, 14, 36, 84)
151	(4, 13, 85, 340)	152	(2, 7, 18, 189)	153	(4, 4, 102, 187)
154	(1, 2, 24, 72)	155	(8, 28, 315, 585)	156	(6, 42, 95, 627)
157	(1, 14, 25, 100)	158	(5, 11, 192, 320)	159	(3, 10, 140, 204)
160	(2, 49, 54, 189)	161	(5, 9, 210, 280)	162	(1, 2, 8, 88)
163	(5, 195, 256, 312)	164	(1, 6, 15, 110)	165	(3, 55, 84, 308)
166	(1, 21, 66, 66)	167	(5, 12, 165, 390)	168	(2, 7, 54, 189)
169	(1, 4, 25, 100)	170	(11, 15, 352, 672)	171	(2, 3, 75, 120)
172	(1, 24, 45, 90)	173	(4, 20, 27, 459)	174	(3, 14, 19, 342)
175	(2, 24, 136, 153)	176	(3, 25, 207, 225)	177	(6, 63, 315, 560)
178	(8, 24, 55, 870)	179	(12, 165, 590, 1180)	180	(1, 8, 56, 91)
181	(1, 4, 30, 105)	182	(1, 5, 18, 120)	183	(7, 160, 240, 777)
184	(4, 6, 75, 340)	185	(1, 18, 76, 76)	186	(2, 35, 95, 210)
187	(2, 56, 104, 189)	188	(7, 217, 264, 744)	189	(5, 195, 312, 384)
190	(2, 5, 18, 225)	191	(5, 36, 369, 410)	192	(3, 42, 200, 280)
193	(4, 35, 40, 553)	194	(12, 35, 188, 1410)	195	(4, 40, 264, 385)
196	$*$	197	(1, 7, 48, 112)	198	(9, 34, 45, 1122)
199	(2, 15, 68, 255)	200	$*$	201	(5, 20, 84, 654)
202	(1, 4, 14, 133)	203	(1, 10, 52, 117)	204	(1, 10, 39, 130)
205	(1, 1, 28, 70)	206	(3, 8, 165, 264)	207	(1, 5, 42, 120)
208	(2, 54, 147, 189)	209	(4, 189, 297, 308)	210	(5, 6, 77, 462)
211	(2, 15, 63, 280)	212	(1, 1, 15, 85)	213	(1, 10, 13, 156)
214	(2, 5, 42, 245)	215	(2, 27, 147, 216)	216	(3, 28, 69, 460)
217	(1, 8, 72, 108)	218	(1, 15, 16, 160)	219	(3, 7, 140, 300)
220	(1, 6, 13, 156)	221	(4, 184, 312, 345)	222	(5, 28, 396, 495)
223	(1, 4, 70, 100)	224	(3, 5, 42, 350)	225	(1, 20, 84, 105)
226	(10, 45, 792, 968)	227	(1, 13, 34, 156)	228	(1, 10, 22, 165)
229	(2, 12, 140, 231)	230	(1, 2, 42, 105)	231	(3, 8, 220, 264)
232	(1, 4, 13, 156)	233	(13, 405, 660, 1782)	234	(1, 48, 70, 105)
235	(4, 210, 245, 441)	236	(4, 100, 259, 525)	237	(3, 35, 90, 504)
238	(2, 10, 13, 325)	239	(4, 135, 351, 420)	240	(2, 9, 10, 315)
241	(1, 40, 69, 120)	242	(1, 5, 72, 120)	243	(5, 35, 88, 880)
244	(4, 21, 175, 600)	245	(1, 9, 20, 180)	246	(2, 55, 90, 315)
247	(5, 22, 341, 620)	248	(1, 18, 38, 171)	249	(5, 12, 352, 495)
250	(2, 5, 98, 245)	251	(2, 21, 105, 320)	252	(1, 14, 84, 132)
253	(1, 6, 20, 180)	254	(3, 114, 247, 364)	255	(4, 39, 87, 754)
256	$*$	257	(6, 287, 364, 819)	258	(5, 36, 246, 820)
259	(2, 11, 104, 312)	260	$*$	261	(3, 40, 42, 595)
262	(1, 7, 48, 168)	263	(3, 184, 228, 345)	264	(1, 35, 90, 126)
265	(5, 231, 420, 616)	266	(3, 195, 286, 286)	267	(3, 16, 33, 572)
268	(13, 15, 336, 1456)	269	(1, 20, 105, 126)	270	(3, 39, 98, 588)
271	(4, 45, 441, 490)	272	(6, 9, 34, 833)	273	(3, 115, 204, 460)
274	(1, 28, 91, 140)	275	(1, 3, 28, 168)	276	(2, 76, 165, 285)
277	(191, 836, 1463, 36290)	278	(10, 21, 360, 1449)	279	(7, 380, 570, 924)
280	(1, 3, 30, 170)	281	(10, 27, 80, 1755)	282	(6, 35, 259, 1110)

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Table A.1. (*Continued*)

n	(w, x, y, z)	n	(w, x, y, z)	n	(w, x, y, z)
283	(1, 3, 48, 156)	284	(3, 5, 96, 416)	285	(3, 84, 290, 435)
286	(6, 275, 555, 814)	287	(1, 9, 44, 198)	288	(1, 8, 18, 216)
289	(1, 12, 22, 220)	290	(3, 10, 91, 546)	291	(11, 200, 300, 2409)
292	(5, 8, 187, 680)	293	(3, 35, 380, 380)	294	(1, 8, 27, 216)
295	(1, 10, 88, 165)	296	(4, 7, 189, 540)	297	(2, 33, 253, 264)
298	(8, 170, 561, 1496)	299	(3, 220, 316, 330)	300	(2, 78, 91, 399)

Appendix B. Further Tables

We computed solutions of the title equation for $n = 4m^2$, $m \equiv 2 \pmod 4$, in the range $n < 20000$, and found solutions in all cases except $n = 10000$ and $n = 15376$; see Table B.1. Further, for $n = 4m^2 + 4$, $m \equiv 2 \pmod 4$, we were able to find solutions in all cases where $n < 20000$; see Table B.2.

Table B.1. Solutions of $(w+x+y+z)(1/w+1/x+1/y+1/z) = 4m^2, m \equiv 2 \pmod 4$.

n	m	(w, x, y, z)	n	m	(w, x, y, z)
144	6	(1, 21, 33, 77)	400	10	(3, 39, 299, 759)
784	14	(1, 33, 209, 513)	1296	18	(47, 55, 1095, 30879)
1936	22	(17, 1813, 2205, 28305)	2704	26	(3, 651, 2415, 4991)
3600	30	(45, 133, 3605, 116109)	4624	34	(1, 25, 169, 4225)
5776	38	(1, 81, 1325, 4293)	7056	42	(1235, 2639, 735315, 5189223)
8464	46	(1, 121, 385, 7865)	10000	50	
11664	54	(5, 561, 4245, 52921)	13456	58	(13, 16245, 53361, 105105)
15376	62		17424	66	(65, 4305, 5265, 1092609)
19600	70	(9, 589, 1833, 170469)			

Table B.2. Solutions of $(w + x + y + z)(1/w + 1/x + 1/y + 1/z) = 4m^2 + 4,\ m \equiv 2 \pmod 4$.

n	m	(w, x, y, z)	n	m	(w, x, y, z)
148	6	(5, 7, 13, 325)	404	10	(1, 9, 25, 315)
788	14	(3, 217, 385, 1705)	1300	18	(5, 637, 1615, 4165)
1940	22	(1, 11, 51, 1683)	2708	26	(7, 759, 2479, 15477)
3604	30	(1, 91, 161, 3289)	4628	34	(13, 21, 285, 35815)
5780	38	(5, 29, 1653, 22895)	7060	42	(43, 121, 88451, 135235)
8468	46	(35, 2171, 54275, 234969)	10004	50	(1, 51, 1131, 8619)
11668	54	(25, 41475, 45899, 203931)	13460	58	(55, 189, 70455, 502335)
15380	62	(1, 3219, 4995, 7155)	17428	66	(27, 125307, 155601, 189371)
19604	70	(123, 459, 2425, 1825443)			

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