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An interesting quartic surface, everywhere locally solvable, with cubic point but no global point

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Abstract. There seem few examples in the literature of quartic surfaces defined over $\mathbb{Q}$ that are everywhere locally solvable, yet which have no global point. It is a delicate question as to whether such surfaces can possess points defined over an odd-degree number field, and to our knowledge no previous example is known. We give here an example of such a diagonal quartic surface which contains a point defined over a cubic extension field (and it follows that there exist number fields of every odd degree greater than 1 in which the surface has points). This surface is one member of a more general family of surfaces, each of which is also everywhere locally solvable but with no rational point.

1. Introduction

Let Γ be an irreducible algebraic variety of degree d in projective space $\mathbb{P}^n$ over a field k. Suppose K is a finite extension of k with $[K : k]$ prime to d. If Γ has a point defined over K, then does Γ necessarily have a point defined over k? This is a question much studied in the literature. Several instances are known where the answer is positive, for example, when Γ is a quadric in $\mathbb{P}^n$ (Springer [14]), or a cubic plane curve (Poincaré [12]), or the intersection of two quadrics (Amer [1], Brumer [7]). Examples are also known where the answer is false, for instance, an intersection of three quadrics in $\mathbb{P}^2$ possessing a zero in a cubic extension of $\mathbb{Q}$, but having no point in $\mathbb{Q}$ (Pfister [11]). Cassels and Swinnerton-Dyer have conjectured that if Γ is a cubic hypersurface in $\mathbb{P}^n$, then a point in K, where $[K : k]$ is prime to 3, implies a point in k; this

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remains unresolved, although in the case of cubic surfaces, Coray [8] has shown that the existence of a point in a field K, with $[K : \mathbb{Q}]$ prime to 3, implies the existence of a point in a field L with $[L : \mathbb{Q}] = 1, 4, 10$. He also gives in this paper an example of a quartic curve over $\mathbb{Q}$ with no rational point (essentially for trivial reasons, because it has no point in $\mathbb{Q}_5$), yet with a point in a cubic extension of $\mathbb{Q}$. Bremner–Lewis–Morton [2] remove the trivial obstruction to existence of a rational point by giving examples of quartic curves of type $\Gamma : ax^4 + by^4 + cz^4 = 0$ which are everywhere locally solvable, have no rational point, yet have a point in a cubic extension of $\mathbb{Q}$. The situation for surfaces, of degree exceeding 3, seems less treated in the literature.

Creutz [10] gives the example of the bielliptic surface

$$(x^2+1)y^2 = (x^2+2)z^2 = 3(t^4 - 54t^2 - 117t - 243),$$

shown by Skorobogatov [13] to have no rational points, yet possessing a point over the field $\mathbb{Q}(\theta)$, $\theta^3 + \theta + 1 = 0$. Here, we focus on surfaces of degree 4. Our goal was to discover a quartic surface over $\mathbb{Q}$, everywhere locally solvable, with no rational point, yet which possesses a point defined over a cubic extension of $\mathbb{Q}$.

The first attempt at discovery of such a surface was to scour the literature for examples of quartic surfaces over $\mathbb{Q}$, everywhere locally solvable, but with no rational point. We could only find such examples arising in the context of diagonal quartic surfaces, which are much studied, particularly by Bright and Swinnerton-Dyer. The search turned up surprisingly few explicit examples, specifically, the surfaces

$$2X_0^4 + 9X_1^4 = 6X_2^4 + 12X_3^4, \qquad 4X_0^4 + 9X_1^4 = 8X_2^4 + 8X_3^4,$$

and the family

$$X_0^4 + 4X_1^4 = d(X_2^4 + X_3^4),$$

where $d > 0$, $d \equiv 2 \bmod 16$, no prime $p \equiv 3 \bmod 4$ divides d, no prime $p \equiv 5 \bmod 8$ divides d to an odd power, and $r \equiv \pm 3 \bmod 8$ where $d = r^2 + s^2$ (see Swinnerton-Dyer [15]); and

$$X_0^4 + X_1^4 = 6X_2^4 + 12X_3^4, \quad X_0^4 + 47X_1^4 = 103X_2^4 + 17 \cdot 47 \cdot 103X_3^4 \qquad (1)$$

(see Example 2.3.1 of Bright [4], and Proposition 3.3 of Bright [3], respectively). However, finding points on these surfaces over any odd degree extension field seems a difficult problem. A naive search for points on these surfaces over cubic fields was unsuccessful in each instance. The second surface at (1) is particularly interesting, because Bright (see [5, Remark 5.20]) proves that there is no Brauer–Manin obstruction to the existence of a rational 0-cycle of degree 1 on the surface.

Accordingly, we changed approach. We first constructed quartic surfaces

$$aX_0^4 + bX_1^4 + cX_2^4 + dX_3^4 = 0 \tag{2}$$

that contain a point over a given cubic extension. This list was winnowed down by demanding the surface be everywhere locally solvable, and was further reduced by demanding the surface contain no rational point with height at most 500. This produced a relatively long list of surfaces (2). It was hoped to show non-existence of rational points on one of these surfaces. In general, this in itself is a difficult Diophantine problem. Our hope lay in finding a surface (2) on the list for which $abcd$ was a perfect square, so that we could try to use ideas from Swinnerton-Dyer [15] to show non-existence of rational points. By sheer luck, our final list of surfaces contained exactly one example of a surface (2) satisfying $abcd = \square$; specifically,

$$V:\ X_0^4 + 7X_1^4 = 14X_2^4 + 18X_3^4. \tag{3}$$

There is the cubic point on V given by

$$(X_0, X_1, X_2, X_3) = (2\theta^2 + 2\theta,\ 2\theta,\ \theta^2 + 1,\ \theta^2 - 1), \quad \theta^3 + \theta^2 - 1 = 0. \tag{4}$$

The non-existence of a rational point on V is an example of a Brauer-Manin obstruction. Our proof of non-existence uses the following principal idea, based on arguments of Swinnerton-Dyer [15]. The existence of a rational point on V induces an elliptic fibration of V, expressing V as the intersection of two parameterized quadrics. The intersection may be represented in different ways; and local solvability conditions in terms of a Hilbert symbol play these intersections against each other to result in a contradiction.

In the following section, we consider a family of surfaces which contains V; and show that each member of this family of K3 surfaces is everywhere locally solvable, but has no rational points.

2. The family of surfaces

Let P, Q be coprime squarefree integers such that every prime factor of PQ is congruent to 1 mod 24. Further, suppose that if $p \mid P$ is prime, then

$2Q^2$ is a fourth power mod p, and if $q \mid Q$ is prime, then $-7P^2$ is a fourth power mod q. There are infinitely many pairs of integers P, Q satisfying these conditions; indeed, infinitely many with $Q = 1$. For by a result of Weber [16] (see also Cox [9, Theorem 9.12]), there exist infinitely many primes of the form $9u^2 + 64v^2$. Such primes P are known from a theorem of Dirichlet to have 2 as a biquadratic residue; and it is clear that $P \equiv 1 \bmod 24$.

Theorem 2.1. *The surface*

$$V_{PQ}: \; X_0^4 + 7P^2X_1^4 = 14P^2Q^2X_2^4 + 18Q^2X_3^4 \tag{5}$$

is (i) *everywhere locally solvable, and* (ii) *has no points defined over* $\mathbb{Q}$.

Proof. (i) It suffices to show local solvability at $p = 2, 3, 5, 7$, and at $p \mid PQ$ (see Bright [6, Lemma 5.2]):

- in $\mathbb{Q}_2$, we have the point $(x_0, x_1, x_2, x_3) = (0, 0, 3, \sqrt[4]{-63}\sqrt{P})$;
- in $\mathbb{Q}_3$, $(\sqrt[4]{7(2Q^2-1)}\sqrt{P}, 1, 1, 0)$;
- in $\mathbb{Q}_5$, $(0, \sqrt[4]{2Q^2(7P^2+9)/(7P^2)}, 1, 1)$ when $(P^2, Q^2) \equiv (1, 1) \bmod 5$, $(\sqrt[4]{14P^2Q^2 + 18Q^2 - 7P^2}, 1, 1, 1)$ when $(P^2, Q^2) \equiv (1, 4)$ or $(4, 1) \bmod 5$, $(\sqrt[4]{2Q^2(7P^2+9)}, 0, 1, 1)$ when $(P^2, Q^2) \equiv (4, 4) \bmod 5$;
- in $\mathbb{Q}_7$, $(\sqrt[4]{18}\sqrt{\left(\frac{Q}{7}\right)Q}, 0, 0, 1)$;
- in $\mathbb{Q}_p$, $p \mid P$: $(\sqrt[4]{2Q^2}\sqrt{3}, 0, 0, 1)$;
- in $\mathbb{Q}_q$, $q \mid Q$: $(\sqrt[4]{-7P^2}, 1, 0, 0)$.

(ii) We are motivated by Swinnerton-Dyer [15]. The equation for V_{PQ} at (5) may be written in the form

$$\begin{aligned} &7(X_0^2 + PX_1^2 - 4PQX_2^2)(X_0^2 + PX_1^2 + 4PQX_2^2) \\ &\quad + (X_0^2 - 7PX_1^2 + 12QX_3^2)(X_0^2 - 7PX_1^2 - 12QX_3^2) = 0, \end{aligned}$$

which implies the elliptic fibration

$$\begin{aligned} u(X_0^2 - 7PX_1^2 + 12QX_3^2) + 7v(X_0^2 + PX_1^2 - 4PQX_2^2) &= 0, \\ u(X_0^2 + PX_1^2 + 4PQX_2^2) - v(X_0^2 - 7PX_1^2 - 12QX_3^2) &= 0 \end{aligned} \tag{6}$$

for a parameter $u : v$. We have, on eliminating X_3, X_2, X_1, X_0, respectively:

$$(u^2-2uv-7v^2)X_0^2+(u^2+14uv-7v^2)PX_1^2+4(u^2+7v^2)PQX_2^2=0, \tag{7}$$
$$(u^2+14uv-7v^2)X_0^2-7(u^2-2uv-7v^2)PX_1^2+12(u^2+7v^2)QX_3^2=0, \tag{8}$$
$$2(u^2+7v^2)X_0^2+7(u^2-2uv-7v^2)PQX_2^2+3(u^2+14uv-7v^2)QX_3^2=0, \tag{9}$$
$$-2(u^2+7v^2)PX_1^2-(u^2+14uv-7v^2)PQX_2^2+3(u^2-2uv-7v^2)QX_3^2=0. \tag{10}$$

We now assume the existence of an integer point (x_0, x_1, x_2, x_3) on V_{PQ}, where without loss of generality, $\gcd(x_0, x_1, x_2, x_3) = 1$. By abuse of notation, we assume henceforth that the corresponding rational parameter at (6) is given by $u : v$, $u, v \in \mathbb{Z}$, $\gcd(u, v) = 1$. Set

$$A = u^2 - 2uv - 7v^2, \qquad B = u^2 + 14uv - 7v^2, \qquad C = u^2 + 7v^2,$$

so that $ABC \neq 0$, and the system (7)–(10) becomes

$$Ax_0^2 + BPx_1^2 + 4CPQx_2^2 = 0, \tag{11}$$
$$Bx_0^2 - 7APx_1^2 + 12CQx_3^2 = 0, \tag{12}$$
$$2Cx_0^2 + 7APQx_2^2 + 3BQx_3^2 = 0, \tag{13}$$
$$-2CPx_1^2 - BPQx_2^2 + 3AQx_3^2 = 0. \tag{14}$$

It is straightforward to verify that any prime dividing two of A, B, C can only equal 2 or 7. Write (11) in the form $-ABPx_0^2 - BCQ(2Px_2)^2 = (BPx_1)^2$, so that for every prime p, the Hilbert symbol

$$(-ABP, -BCQ)_p = 1. \tag{15}$$

Let $S = \{2, 3, 7, \infty\} \cup \{p \text{ prime} : p \mid PQ\}$.

Case 1. Suppose $p \notin S$ and $p \mid C$; then p is odd and $p \nmid ABPQ$, so that

$$(-ABP, -BQ)_p = 1.$$

Hence from (15)

$$(-ABP, C)_p = 1 \quad \text{for all } p \notin S, p \mid C. \tag{16}$$

Similarly, writing (14) in the form $3ABP(Qx_3)^2 - 2BCQ(Px_1)^2 = (BPQx_2)^2$, it follows that $(3ABP, -2BCQ)_p = 1$ for all primes p. Further, $p \notin S$ and $p \mid C$ implies $(3ABP, -2BQ)_p = 1$, so that

$$(3ABP, C)_p = 1 \quad \text{for all } p \notin S, p \mid C. \tag{17}$$

We deduce from (16), (17), that

$$(-3, C)_p = 1 \quad \text{for all } p \notin S, p \mid C. \tag{18}$$

Case 2. Suppose $p \notin S$ and $p \nmid C$. Then both -3 and C are units in $\mathbb{Z}_p$, so that

$$(-3, C)_p = 1 \quad \text{for all } p \notin S, p \nmid C.$$

Together with (18), and using the product formula for the Hilbert symbol, it follows that

$$\prod_{p \in S} (-3, C)_p = \prod_{p \notin S} (-3, C)_p = 1. \tag{19}$$

Now $-3 \in (\mathbb{Q}_7^*)^2$, so that $(-3, C)_7 = 1$; and $C = u^2 + 7v^2 > 0$, so that $(-3, C)_\infty = 1$. Further, for p prime with $p \mid PQ$, then $p \equiv 1 \bmod 6$, so that $-3 \in (\mathbb{Q}_p^*)^2$, and $(-3, C)_p = 1$. Equation (19) now becomes

$$(-3, u^2 + 7v^2)_2 \, (-3, u^2 + 7v^2)_3 = 1. \tag{20}$$

Certainly, $3 \mid uv$. For otherwise, (8) implies $uv(x_0^2 + x_1^2) \equiv 0 \pmod 3$, so that $3 \mid x_0$, $3 \mid x_1$. Then from (7), $3 \mid x_2$, and finally from (9), $3 \mid x_3$. Consequently, since $\gcd(u, v) = 1$, we have $u^2 + 7v^2 \equiv 1 \pmod 3$, and $u^2 + 7v^2 \in (\mathbb{Q}_3^*)^2$. Thus $(-3, u^2 + 7v^2)_3 = 1$, and so from (20),

$$(-3, u^2 + 7v^2)_2 = 1. \tag{21}$$

To deal with the 2-adic symbol, we prove the following lemma.

Lemma 2.2. *Write $C = u^2 + 7v^2 = 2^m a$, where a is odd. Then m is odd.*

Proof. Suppose $m = 2n$ is *even*. If $u+v$ is odd, then (8) implies $x_0^2 + x_1^2 \equiv 0 \pmod 4$, so that $2 \mid x_0$, $2 \mid x_1$. Then (10) implies $-x_2^2 + 3x_3^2 \equiv 0 \pmod 4$, so $2 \mid x_2$, $2 \mid x_3$. We deduce u,v are both odd (and hence $m \geq 3$).

Case (i). If $4 \nmid u - v$, write $u - v = 2\alpha$, with α odd. Then $A = u^2 - 2uv - 7v^2 = 4(\alpha^2 - 2v^2) \equiv -4 \bmod 32$, so that $A = -\beta^2$, $\beta \in \mathbb{Q}_2^*$. Similarly, $B = u^2 + 14uv - 7v^2 = 4(\alpha^2 + 4uv - 2v^2) = 4(8k+3)$, $k \in \mathbb{Z}$. From equation (11),

$$4(8k+3)Px_1^2 + 2^{2n+2}aPQx_2^2 = \beta^2 x_0^2,$$

so that

$$1 = (4(8k+3)P, 2^{2n+2}aPQ)_2 = (8k+3, a)_2, \tag{22}$$

on observing P, Q are squares in $\mathbb{Q}_2^*$. However, from (12),

$$4(8k+3)x_0^2 + 12 \cdot 2^{2n} aQx_3^2 = -7\beta^2 Px_1^2,$$

from which, on observing $-7P$ is a square in $\mathbb{Q}_2^*$,

$$1 = (4(8k+3), 12 \cdot 2^{2n} aQ)_2 = (8k+3, 3a)_2.$$

Together with (22), we deduce $1 = (8k+3, 3)_2 = (-1)^{\frac{8k+3-1}{2}\frac{3-1}{2}} = -1$, which is impossible.

Case (ii). If $4 \mid u-v$, write $u-v = 4l$. Then $C = u^2+7v^2 = 8(v^2+vl+2l^2) = 2^{2n}a$, so that l is odd. Further, $A = u^2 - 2uv - 7v^2 = 8(2l^2 - v^2) \equiv 8 \bmod 64$, so that $A = 8\gamma^2$, $\gamma \in \mathbb{Q}_2^*$. Also, $B = u^2+14uv-7v^2 = 8(v^2+8lv+2l^2) = 8(8h+3)$, $h \in \mathbb{Z}$. Equation (11) becomes

$$-(8h+3)Px_1^2 - 2^{2n-1}aPQx_2^2 = \gamma^2 x_0^2,$$

from which

$$\begin{aligned} 1 &= (-(8h+3)P, -2^{2n-1}aPQ)_2 = (-8h-3, -2a)_2 \\ &= (-1)^{\frac{-8h-3-1}{2}\frac{-a-1}{2}+\frac{(8h+3)^2-1}{8}} = (-1)^{8h^2+6h+1} = -1, \end{aligned}$$

a contradiction. Necessarily, therefore, m is odd. $\square$

We can now finish the proof of Theorem 2.1.

We have

$$(-3, u^2+7v^2)_2 = (-3, 2^m a)_2 = (-1)^{\frac{-3-1}{2}\frac{a-1}{2}+m\frac{(-3)^2-1}{8}} = (-1)^m = -1,$$

contradicting (21). $\square$

Corollary 2.3. *Let $n \geq 3$ be an odd integer. There exists a number field K of degree n over $\mathbb{Q}$ such that the surface V in (3) contains points in K.*

Proof. The cubic point at (4) lies on the plane $X_0 = X_1 + X_2 + X_3$ which cuts V in an absolutely irreducible plane quartic curve $\mathcal{C}$. The curve $\mathcal{C}$ of genus 3 accordingly has points in a cubic field, giving rise to a positive divisor of points on $\mathcal{C}$ of degree 3. By Theorem 6.1 of Coray [8], $\mathcal{C}$ contains positive divisors of every odd degree d at least 3. An application of the Hilbert Irreducibility Theorem (see Coray [8, Section 2]) implies the existence on $\mathcal{C}$ of an irreducible $\mathbb{Q}$-rational 0-cycle of degree d; and the corollary now follows. $\square$

Remark 2.4. A referee points out that there certainly exist quartic surfaces over *finite* fields with no global point (for example, the Fermat quartic $X_0^4+X_1^4+X_2^4+X_3^4 = 0$ over $\mathbb{F}_5$), with the property that they have points over any extension of the ground field of sufficiently large degree.

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