$y^2 = x^6 + k,\ k \in \{-39, -47\}$

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Abstract. The aim of this paper is to solve the equation $y^2 = x^6 + k$ in rational numbers with $k \in \{-39, -47\}$. These are the two unsolved cases for integers k in the range $|k| \leq 50$

1. Introduction

In their paper, Brenner and Tzanakis [2] studied the equation $y^2 = x^6 + k$ in rational numbers where k is an integer in the range $|k| \leq 50$. They solved all the equations except $k = -39$ and $k = -47$. The main approach used by Bremner and Tzanakis is the elliptic curve Chabauty method. In this paper, we shall solve the equation $y^2 = x^6 + k$ with $k = -39$ or $k = -47$. For $k = -39$, we shall present two approaches which might be applicable to other values of k. For $k = -47$, only one approach is presented. The main tools here are the elliptic curve Chabauty method and algebraic number theory. In summary, we shall prove:

Theorem 1.1. *The only rational solutions (x, y) to the equation*

$$y^2 = x^6 - 39$$

are $(\pm 2, \pm 5)$

Theorem 1.2. *The only rational solutions (x, y) to the equation*

$$y^2 = x^6 - 47$$

are $(\pm\frac{63}{10}, \pm\frac{249953}{10^3})$.

2. Equation $y^2 = x^6 - 39$

In this section we shall present the proof of Theorem 1.1.

Proof. The equation $y^2 = x^6 - 39$ is equivalent to

$$Y^2 = X^6 - 39Z^6, \tag{2.1}$$

where X, Y, Z are coprime integers. We have

$$(X^3 - Y)(X^3 + Y) = 39Y^2.$$

Let $d = \gcd(X^3 - Y, X^3 + Y)$. Then $d|\gcd(2X^3, 2Y) = 2$. We can choose the sign of Y such that $13|X^3 + Y$.

Case $d = 1$: we have

$$X^3 + Y = 39V^6, \quad X^3 - Y = U^6, \quad \gcd(U, V) = 1,$$

or

$$X^3 + Y = 13V^6, \quad X^3 - Y = 3U^6, \quad \gcd(U, V) = 1.$$

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So

$$2X^3 = 39V^6 + U^6 \quad \text{or} \quad 2X^3 = 13V^6 + 3U^6, \quad \gcd(U, V) = 1.$$

In the former case, we have $3 \nmid U$. So $U^6 \equiv 1 \mod 9$, hence $2X^3 \equiv 3V^6 + 1 \mod 9$. Thus $X^3 \equiv -1 \mod 3$, so $X \equiv -1 \mod 3$. Therefore $X^3 \equiv -1 \mod 9$. So $V^6 + 1 \equiv 0 \mod 3$, impossible.

In the latter case, we have

$$2X^3 = 13V^6 + 3U^6, \quad \gcd(U, V) = 1. \tag{2.2}$$

We shall deal with this case later.

Case $d = 2$: we have

$$X^3 + Y = 2 \cdot 39V^6, \quad X^3 - Y = 2^5 \cdot U^6, \quad \gcd(U, V) = 1,$$
$$X^3 + Y = 2^5 \cdot 39V^6, \quad X^3 - Y = 2 \cdot U^6, \quad \gcd(U, V) = 1,$$
$$X^3 + Y = 2 \cdot 13V^6, \quad X^3 - Y = 2^5 \cdot 3U^6, \quad \gcd(U, V) = 1,$$
$$X^3 + Y = 2^5 \cdot 13V^6, \quad X^3 - Y = 2 \cdot 3U^6, \quad \gcd(U, V) = 1.$$

This gives

$$X^3 = 39V^6 + 16U^6,$$
$$X^3 = 624V^6 + U^6,$$
$$X^3 = 13V^6 + 48U^6,$$
$$X^3 = 208V^6 + 3U^6.$$

The first equation: $\pm 1, \pm 5 \equiv 3U^6 \equiv \pm 3 \mod 13$, impossible.
The third equation: $\pm 1, \pm 5 \equiv \pm 4 \mod 13$, impossible.
The fourth equation: $\pm 1, \pm 5 \equiv \pm 3 \mod 13$, impossible.
There remains the second equation:

$$X^3 = 624V^6 + U^6, \quad \gcd(U, V) = 1.$$

This gives

$$(624(X/U^2))^3 = (624(V^3/U^3))^2 + 624^3.$$

The elliptic curve $y^2 = x^3 - 624^3$ has rank 0, so $X^3 = 624V^6 + U^6$ only has trivial solutions.

We only need to deal with the case (2.2)

$$2X^3 = 3U^6 + 13V^6, \quad \gcd(U, V) = 1.$$

Observe that $2 | X$ and $2 \nmid U, V$.

Solution 1: Let $K = \mathbb{Q}(\theta)$, where $\theta = \sqrt[3]{39}$. K has the ring of integers $\mathcal{O}_K = \mathbb{Z}[\theta]$ and a fundamental unit $\epsilon = 2\theta^2 - 23$ of norm 1.

Lemma 2.1. *Consider the elliptic curve*

$$E : v^2 = u^3 - 39,$$

let ϕ be a map $E(\mathbb{Q}) \to K^*/(K^*)^2$ given by

$$\phi(u, v) = u - \theta \mod (K^*)^2,$$
$$\phi(\infty) = (K^*)^2.$$

Then ϕ is a group homomorphism with the kernel $2E(\mathbb{Q})$.

Proof. This is the standard 2-descent. See Silverman [5]. □

We have
$$E(\mathbb{Q}) = \mathbb{Z}(10,31) \oplus \mathbb{Z}(4,5).$$
Because $(X^2/Z^2, Y/Z^3) \in E(\mathbb{Q})$, Lemma 2.1 implies
$$(X^2 - \theta Z^2) \equiv \alpha \mod (K^*)^2,$$
where $\alpha \in \{1, 4-\theta, 10-\theta, (4-\theta)(10-\theta)\}$.
Because $10 - \theta = \epsilon(3\theta^2 + 10\theta + 34)^2$, we have the following cases:

Case 1: $X^2 - \theta Z^2 \in K^2$.
Because $X^2 - \theta Z^2 \in \mathbb{Z}[\theta] = \mathcal{O}_K$, we have
$$X^2 - \theta Z^2 = (a + b\theta + c\theta^2)^2,$$
where $a, b, c \in \mathbb{Z}$. Comparing coefficients of $\theta^0, \theta, \theta^2$ gives:
$$\begin{cases} X^2 = a^2 + 78bc, \\ Z^2 = -2ab - 39c^2, \\ 0 = 2ac + b^2. \end{cases}$$
From $\gcd(X, Z) = 1$, we have $\gcd(a, b, c) = 1$. Because $2|X$, from the first and the third equations, we have $2|a,\ b$. Thus $2 \nmid c$. Let $a = 2a_1$, $b = 2b_1$. Then
$$\begin{cases} (X/2)^2 = a_1^2 + 39b_1c, \\ Z^2 = -8a_1b_1 - 39c^2, \\ 0 = a_1c + b_1^2. \end{cases}$$
Since $\gcd(a, b, c) = 1$, the third equation implies $\gcd(a_1, c) = 1$. Hence $\exists\, r, s \in \mathbb{Z}$, $r > 0$ such that
$$a_1 = r^2, \quad c = -s^2, \quad b_1 = -rs, \quad \gcd(r, s) = 1, \tag{2.3}$$
or
$$a_1 = -r^2, \quad c = s^2, \quad b_1 = -rs, \quad \gcd(r, s) = 1. \tag{2.4}$$
Case (2.3) gives
$$(X/2)^2 = r(r^3 - 39s^3),$$
$$Z^2 = s(8r^3 - 39s^3).$$
Because $\gcd(X, Z) = 1$, we have $\gcd(r, 39) = \gcd(s, 2) = 1$. Hence $\gcd(r, r^3 - 39s^3) = \gcd(s, 8r^3 - 39s^3) = 1$. Because $r > 0$, we have $8r^3 - 39s^3 > r^3 - 39s^3 > 0$. Thus $s > 0$. It follows that
$$r = A^2, \quad r^3 - 39s^3 = C^2, \quad X = \pm AC,$$
$$s = B^2, \quad 8r^3 - 39s^3 = D^2, \quad Z = \pm BD.$$
Therefore $D^2 = 8A^6 - 39B^6$. So $D^2 + A^6 \equiv 0 \mod 3$. Hence $A \equiv D \equiv 0 \mod 3$. Thus $3|X,\ Z$, a contradiction.
Case (2.4) gives
$$(X/2)^2 = r(r^3 - 39s^3),$$
$$Z^2 = -s(8r^3 + 39s^3).$$
We have $\gcd(r, 39) = \gcd(s, 2) = 1$. Because $r > 0$, if $s > 0$, then $Z^2 = -s(8r^3 + 39s^3) < 0$, impossible. Therefore $s < 0$. Thus
$$r = A^2, \quad r^3 - 39s^3 = C^2, \quad X = \pm AC,$$
$$s = -B^2, \quad 8r^3 + 39s^3 = D^2, \quad Z = \pm BD.$$

Thus $D^2 = 8A^6 - 39B^6$. So $D^2 + A^6 \equiv 0 \mod 3$. Therefore $A \equiv D \equiv 0 \mod 3$. Hence $3|X,\ Z$, a contradiction.

Case 2: $(X^2 - \theta Z^2) \in \epsilon K^2$.

Because ϵ is a unit and $X^2 - \theta Z^2 \in \mathcal{O}_K$, we have

$$X^2 - \theta Z^2 = (2\theta^2 - 23)(a + b\theta + c\theta^2)^2,$$

where $a, b, c \in \mathbb{Z}$. Comparing the coefficients of $\theta^0, \theta, \theta^2$ gives

$$\begin{cases} X^2 = -23a^2 + 156ab - 1794bc + 3042c^2, \\ Z^2 = 46ab - 156ac - 78b^2 + 897c^2, \\ 0 = 2a^2 - 46ac - 23b^2 + 156bc. \end{cases}$$

Because $\gcd(X, Z) = 1$, we have $\gcd(a, b, c) = 1$. From the third equation, we have $2|b$. $2|X$. Thus the first equation implies $2|a$. Hence $2 \nmid c$. The first equation gives

$$X^2 \equiv 2c^2 \equiv 2 \mod 4,$$

impossible.

Case 3: $X^2 - \theta Z^2 \in \epsilon(4 - \theta)K^2$.

Let

$$X^2 - \theta Z^2 = \epsilon(4 - \theta)(\frac{a + b\theta + c\theta^2}{n})^2,$$

where $n, a, b, c \in \mathbb{Z}$ and $\gcd(a, b, c) = 1$. Comparing the coefficients of $\theta^0, \theta, \theta^2$ gives

$$\begin{cases} (nX)^2 = -170a^2 + 624ab + 1794ac + 897b^2 - 13260bc + 12168c^2, \\ (nZ)^2 = -23a^2 + 340ab - 624ac - 312b^2 - 1794bc + 6630c^2, \\ 0 = 8a^2 + 46ab - 340ac - 170b^2 + 624bc + 897c^2. \end{cases}$$

From the third equation, we have $2|c$. Because $2|nX$, from the first equation, we have $2|b$. Therefore $2 \nmid a$. Then the first equation gives

$$(nX)^2 \equiv 2a^2 \equiv 2 \mod 4,$$

impossible.

Case 4: $(X^2 - \theta Z^2)(4 - \theta) \in K^2$.

We have $x = X/Z$, $y = Y/Z^3$, $y^2 = (x^2 - \theta)(x^4 + \theta x^2 + \theta^2)$, and $(x^2 - \theta)(4 - \theta) \in K^2$. Thus

$$(4 - \theta)(x^4 + \theta x^2 + \theta^2) \in K^2.$$

Let $(4 - \theta)(x^4 + 4\theta x^2 + \theta^2) = \beta^2$. Then $((4 - \theta)x^2, (4 - \theta)\beta)$ is a point on

$$G : v^2 = u(u^2 + \theta(4 - \theta)u + \theta^2(4 - \theta)^2).$$

We have

$$G(K) = \mathbb{Z}/2\mathbb{Z}(0, 0) \oplus \mathbb{Z}(\frac{4\theta^2 - 39}{4}, \frac{20\theta^2 - 195}{8}).$$

The curve G has rank 1 over K, and $[K : \mathbb{Q}] = 3$.

The first approach is to use the elliptic curve Chabauty method. With the search bound of 350 and the assumption of the Generalized Riemann Hypothesis, PseudoMordellWeil returns "false". The second approach is to use the formal group technique as in Flynn [3] which will almost guarantee the solution when $\text{rank}(G(K)) < [K : \mathbb{Q}]$. If we follow this approach, then the smallest prime that might work is $p = 7$. The order of the generator $(\frac{4\theta^2-39}{4}, \frac{20\theta^2-195}{8})$ in $\mathbb{F}_7(\theta)$ with $\theta^3 - 39 = 0$ is 86. In $G(K)$, we shall need to compute the set $\{m(0, 0) + n(\frac{4\theta^2-39}{4}, \frac{20\theta^2-195}{8}) \colon n = 0, 1, m = -42, -41, ..., m = 43\}$ and then compute the corresponding formal power series, see

Flynn [3] for more details about this approach. This might work, but it shall take too much computation. We will take another approach which might possibly be applicable in case $\operatorname{rank}(G(K)) \geq [K : \mathbb{Q}]$.
We have

$$X^2 - \theta Z^2 = (4-\theta)(a + b\theta + c\theta^2)^2,$$

where $a, b, c \in \mathbb{Q}$. Thus

$$X^2 = 4a^2 - 78ac - 39b^2 + 312bc, \tag{2.5}$$

$$Z^2 = a^2 - 8ab + 78bc - 156c^2, \tag{2.6}$$

$$0 = -2ab + 8ac + 4b^2 - 39c^2. \tag{2.7}$$

If $4c - b = 0$, then from (2.7), we have $4b^2 - 39c^2 = 0$. So $b = c = 0$. Therefore

$$x = \frac{X}{Z} = \pm 2.$$

If $4c - b \neq 0$, then from (2.7), we have $a = \frac{39c^2 - 4b^2}{2(4c-b)}$.
Let $P = 5c$ and $Q = 4c - b$. Then

$$X^2 = \frac{P^4 - 5P^3Q + 24P^2Q^2 - 20PQ^3 - 23Q^4}{Q^2},$$

$$Z^2 = \frac{P^4 - 24P^2Q^2 + 40PQ^3 - 48Q^4}{4Q^2}.$$

Let $P = dp$, $Q = dq$, $X_1 = \frac{qX}{d}$, $Z_1 = \frac{2qZ}{d}$, where $d = \gcd(P, Q)$. Then

$$\begin{aligned} X_1^2 &= p^4 - 5p^3q + 24p^2q^2 - 20pq^3 - 23q^4, \\ Z_1^2 &= p^4 - 24p^2q^2 + 40pq^3 - 48q^4. \end{aligned} \tag{2.8}$$

We have $\gcd(p, q) = 1$ and $X_1, Z_1 \in \mathbb{Z}$.

Lemma 2.2. *In (2.8), we have*

$$\gcd(X_1, 39) = \gcd(Z_1, 13) = \gcd(Z_1, 2) = 1.$$

Proof. First, we show that $2 \nmid Z_1$.
If $q \nmid d$, then $\exists$ a prime $l | q$ such that $l | X_1 = \frac{qX}{d}$. Thus

$$l | p^4 - 5p^3q + 24p^2q^2 - 20pq^3 - 23q^4.$$

Because $l | q$, we have $l | p$. So $l | \gcd(p, q) > 1$, a contradiction. Therefore $q | d$. Thus $X_1 | X$ and $Z_1 | 2Z$. From (2.2), we have $\gcd(U, V) = 1$, $2 | X$ and $2 \nmid Z$. If $2 | Z_1$. Then from

$$Z_1^2 = p^4 - 24p^2q^2 + 40pq^3 - 48q^4,$$

we have $2 | p$. Thus $2 \nmid q$. Hence $2 \nmid X_1$. From $2 | X = (\frac{d}{q})X_1$, we have $2 | \frac{d}{q}$. So $\frac{d}{2q} \in \mathbb{Z}$. Because $2 \nmid Z = (\frac{d}{2q})Z_1$, we have $2 \nmid Z_1$, a contradiction. So $2 \nmid Z_1$.
If $3 | X_1$, then

$$3 | p^4 - 5p^3q + 24p^2q^2 - 20pq^3 - 23q^4.$$

Thus

$$3 | p^4 + q^4 + p^3q + qp^3.$$

Because $\gcd(p, q) = 1$, we have $3 \nmid p, q$. Hence $3 | 2 + 2pq$. So $pq \equiv -1 \mod 3$, thus $p + q \equiv 0 \mod 3$. Therefore

$$Z_1^2 = p^4 - 24p^2q^2 + 40pq^3 - 48q^4 \equiv -3p^4 \mod 9,$$

which is not possible. So $3 \nmid X_1$.
If $13|X_1$, then

$$13|p^4 - 5p^3q + 24p^2q^2 - 20pq^3 - 23q^4.$$

Thus $13|p + 2q$. So

$$Z_1^2 = p^4 - 24p^2q^2 + 40pq^3 - 48q^4 \equiv -39q^4 \mod 13^2,$$

which is not possible. Hence $13 \nmid X_1$.
If $13|Z_1$, then

$$13|p^4 - 24p^2q^2 + 40pq^3 - 48q^4.$$

Thus

$$13|(p + 2q)(p + 7q).$$

If $13|p + 2q$ or $13|p + 7q$, then

$$Z_1^2 = p^4 - 24p^2q^2 + 40pq^3 - 48q^4 \equiv -39q^4 \mod 13^2,$$

which is not possible. So $13 \nmid Z_1$.

□

Let $L = \mathbb{Q}(\phi)$, where ϕ, ~ 2.8502, is the largest real root of $x^4 - 6x^2 - 5x - 3 = 0$. L has class number 1, the ring of integers $\mathcal{O}_L = \mathbb{Z}[\phi]$, and two positve fundamental units $\epsilon_1 = \phi + 2$, $\epsilon_2 = \phi^3 - \phi^2 - \phi - 1$ with $\text{Norm}(\epsilon_1) = \text{Norm}(\epsilon_2) = -1$.
Let

$$F(p, q) = p^4 - 5p^3q + 24p^2q^2 - 20pq^3 - 23q^4,$$
$$G(p, q) = p^4 - 24p^2q^2 + 40pq^3 - 48q^4.$$

Then

$$F(p, q) = (p + (\phi^3 - 7\phi - 5)q)A(p, q),$$
$$G(p, q) = (p + 2\phi q)B(p, q),$$

where

$$A(p, q) = p^3 + (-\phi^3 + 7\phi)p^2q + (4\phi^2 - 5\phi)pq^2 + (4\phi^3 - 5\phi^2 - 12\phi - 5)q^3,$$
$$B(p, q) = p^3 - 2\phi p^2q + (4\phi^2 - 24)pq^2 + (-8\phi^3 + 48\phi + 40)q^3.$$

In $\mathbb{Z}[\phi]$, let

$$p_1 = -2\phi^3 + \phi^2 + 12\phi + 4, \quad p_2 = \phi, \quad p_3 = \phi + 1, \quad q_1 = \phi^3 - 6\phi - 4, \quad q_2 = \phi - 1.$$

Then

$$3 = p_1p_2p_3^3, \quad 13 = q_1q_2^3,$$

$$\text{Norm}(p_1) = 1, \text{Norm}(p_2) = \text{Norm}(p_3) = -3,$$

$$\text{Norm}(q_1) = \text{Norm}(q_2) = -13.$$

We also have

$$\text{Res}(p + 2\phi q, B(p, q)) = -8p_1p_2^5q_2^2,$$
$$\text{Res}(p + (\phi^3 - 7\phi - 5)q, A(p, q)) = (4\phi^3 + 6\phi^2 - 31\phi - 53)p_2p_3^6q_1q_2^3.$$

Because $\gcd(X_1, 39) = \gcd(Z_1, 39) = \gcd(Z_1, 2) = 1$ and $\mathrm{Norm}(4\phi^3+6\phi^2-31\phi-53) = 1$, we have

$$\begin{cases} p + (\phi^3 - 7\phi - 5)q = (-1)^h \epsilon_1^i \epsilon_2^j S^2, \quad p + 2\phi q = (-1)^{h_1} \epsilon_1^{i_1} \epsilon_2^{j_1} T^2, \\ A(p,q) = (-1)^h \epsilon_1^{-i} \epsilon_2^{-j} S_1^2, \quad B(p,q) = (-1)^{h_1} \epsilon_1^{-i_1} \epsilon_2^{-j_1} T_1^2, \end{cases}$$

where $X_1 = SS_1$ and $Z_1 = TT_1$.
Taking norms gives

$$(X_1)^2 = (-1)^{i+j}\,\mathrm{Norm}(S)^2, \quad Z_1^2 = (-1)^{i_1+j_1}\,\mathrm{Norm}(T)^2.$$

Thus $2|i+j$ and $2|i_1+j_1$. Hence $i = j$ and $i_1 = j_1$.
Let $\beta = \epsilon_1\epsilon_2 = \phi^3 + 3\phi^2 + 2\phi + 1 > 0$. Then

$$\begin{cases} p + (\phi^3 - 7\phi - 5)q = (-1)^h \beta^i S^2, \quad p + 2\phi q = (-1)^{h_1} \beta^{i_1} T^2, \\ A(p,q) = (-1)^h \beta^{-i} S_1^2, \quad B(p,q) = (-1)^{h_1} \beta^{-i_1} T_1^2. \end{cases} \tag{2.9}$$

Lemma 2.3. *We have*

$$(p + (\phi^3 - 7\phi - 5)q)(p + 2\phi q) > 0. \tag{2.10}$$

Proof. Equation $F(x,1) = 0$ has 2 real roots

$$x_1 = -\phi^3 + 7\phi + 5 \sim 1.7976,\ x_2 \sim -0.6206.$$

Equation $G(x,1) = 0$ has 2 real roots

$$x_3 = -2\phi \sim -5.7004,\ x_4 \sim 4.1399.$$

We have

$$F(\frac{p}{q}, 1) > 0 \quad \textit{and} \quad G(\frac{p}{q}, 1) > 0.$$

So

$$\frac{p}{q} < x_3 \quad \textit{or} \quad \frac{p}{q} > x_4.$$

Because $x_3 < x_2 < x_1 < x_4$, we have

$$(p + x_1 q)(p + x_3 q) > 0.$$

$\square$

From Lemma 2.3 and (2.9), we have $h = h_1$. So by mapping $(p,q) \mapsto (-p,-q)$, we can assume that $h = h_1 = 0$.

Case $i \neq i_1$:
Because $\phi - 1|\phi^3 - 9\phi - 5$, we have

$$(\phi - 1)|(\phi^3 - 9\phi - 5)q = \beta^i S^2 - \beta^{i_1} T^2.$$

Because $i - i_1 = \pm 1$ and β is a unit, we have

$$\beta S^2 - T^2 \equiv 0 \mod \phi - 1.$$

If $\phi - 1|S$ or $\phi - 1|T$, then $\phi - 1|S,\, T$. Hence $13 = -\mathrm{Norm}(\phi-1)|\,\mathrm{Norm}(S),\, \mathrm{Norm}(T)$. Thus $13|X,\, Z$, impossible. So $\phi - 1 \nmid S,\, T$. Therefore $S^{12} \equiv T^{12} \equiv 1 \mod \phi - 1$ (because $\mathrm{Norm}(\phi - 1) = -13$). Also $\beta \equiv 7 \mod \phi - 1$, therefore

$$0 \equiv \beta^6 S^{12} - T^{12} \equiv 7^6 - 1 \mod \phi - 1.$$

So $13 = -\operatorname{Norm}(\phi - 1) | (7^6 - 1)^4$. But $13 \nmid 7^6 - 1$, so we have a contradiction.

Case $i = i_1$:

If $q \neq 0$, then

$$(p + (\phi^3 - 7\phi - 5)q)(p^3 - 2\phi p^2 q + 4(\phi^2 - 6)pq^2 + 8(-\phi^3 + 6\phi + 5)q^3) = (ST_1)^2,$$

which represents an elliptic curve

$$C\colon v^2 = (u + \gamma)(u^3 - 2\phi u^2 + 4(\phi^2 - 6)u + 8(-\phi^3 + 6\phi + 5)),$$

where $v = (ST_1)/q^2$, $u = p/q$. The minimal cubic model at $(-\gamma, 0)$ is

$$y^2 = x^3 + (-2s^3 + 2s^2 + 10s + 6)x^2 + (-4s^3 + 8s^2 + 12s)x + (1488s^3 + 1776s^2 - 11128s - 17160).$$

The elliptic Chabauty routine in Magma [1] works and returns $u = 69/26$. Hence $(p, q) = (69, 26)$, $(-69, -26)$. This gives no solutions (X_1, Z_1).

Therefore $q = 0$, so $X_1 = \pm 2$ and $Z_1 = \pm 1$. Thus

$$x = \frac{X_1}{Z_1} = \pm 2.$$

So the only rational solutions to $y^2 = x^6 - 39$ are $(x, y) = (\pm 2, \pm 5)$.

Remark 2.4. (i) From the system (2.8), we have a curve

$$F\colon \omega^2 = (\lambda^4 - 5\lambda^3 + 24\lambda^2 - 20\lambda - 23)(\lambda^4 - 24\lambda^2 + 40\lambda - 48), \tag{2.11}$$

where $\omega = \frac{X_1 Z_1}{q^4}$ and $\lambda = \frac{p}{q}$. This curve has genus 3 and the Jacobian rank at most 3. We are unable to compute the Jacobian rank. Computer search reveals no rational points on (2.11). It might be possible to show F has no rational points using the partial descent on hyperelliptic curves as in Siksek and Stoll [4] but we have not proceeded in this way.

(ii) More generally, **Solution 1** gives us an approach to the equation $y^2 = x^6 + k$ in principle. We write $y^2 = x^6 + k$ as $Y^2 = X^6 + kZ^6$, then compute the generators of the MordellWeil group of the elliptic curve $E_k\colon v^2 = x^3 + k$. Using 2-descent as in Lemma 2.1, we shall need to solve a finite number of equations

$$X^2 - \theta Z^2 = (x_i - \theta)(a_i + b_i\theta + c_i\theta^2)^2,$$

where $\theta = k^{1/3}$ and the set $\{(x_i, y_i)\}_i$ is a finite set $a_i, b_i, c_i \in \mathbb{Q}$.

Thus for each i, we have a system of equations:

$$\begin{cases} X^2 = S_0(a_i, b_i, c_i), \\ Z^2 = S_1(a_i, b_i, c_i), \\ 0 = S_3(a_i, b_i, c_i), \end{cases}$$

where S_0, S_1, S_2 are homogenous rational polynomials of degree 2 in a_i, b_i, c_i.

Assume from $S_3(a_i, b_i, c_i) = 0$ that we can solve for one of a_i, b_i, c_i in term of the two other variables. Then from $(XZ)^2 = S_0(a_i, b_i, c_i)S_1(a_i, b_i, c_i)$, we have a genus 3 curve

$$F_i\colon \omega^2 = p_i(\lambda)q_i(\lambda),$$

where $p_i(\lambda), q_i(\lambda)$ are rational polynomials of degree 4. The partial descent method and the Chabauty method might help to find rational points on F_i.

Solution 2: In this section, we shall present another solution to $y^2 = x^6 - 39$. The approach taken here is classical and is applied to the case $k = -47$. We shall start from (2.2)

$$2X^3 = 3U^6 + 13V^6, \quad Z = UV, \quad \gcd(U, V) = 1. \tag{2.12}$$

Observe that U, V are odd and X is even. Let $K = \mathbb{Q}(\theta)$, where $\theta^2 = -39$. The ring of integers is $\mathcal{O}_K = \mathbb{Z}[\frac{1+\theta}{2}]$. The class number is 4. The ideal $(2) = p_{21}p_{22}$, where $p_{21} = (2, \frac{1+\theta}{2})$ and $p_{21}^4 = (\frac{5+\theta}{2})$; the ideal $(3) = p_3^2$, where $p_3 = (3, \theta)$; and $(\theta) = p_3 p_{13}$. We write (2.12) as

$$\frac{(3U^3 + \theta V^3)}{2} \frac{(3U^3 - \theta V^3)}{2} = 12(\frac{X}{2})^3.$$

A common ideal divisor J of the factors on the left divides $(3U^3) = p_3^2(U)^3$ and $p_3p_{13}(V)^3$. J^2 divides $(12(\frac{X}{2})^3) = p_{21}^2p_{22}^2p_3^2(\frac{X}{2})^3$. Certainly, p_3 divides J. Since $J|p_3p_{13}(V)^3$ and $3 \nmid V$, we have $p_3^2 \nmid J$. Further $p_{13} \nmid J$, otherwise $13|X$, impossible. So $J = p_3$.

Since $p_{22}^2|(\frac{3U^3+\theta V^3}{2})$, we have

$$\begin{aligned}(\frac{3U^3 + \theta V^3}{2}) &= p_3p_{22}^2\mathcal{A}^3\\ &= (\frac{3+\theta}{2})\mathcal{A}^3.\end{aligned}$$

It follows that $\mathcal{A}$ is principal. Hence $\mathcal{A} = (A)$ for some element $A \in \mathcal{O}_K$. Further, any unit in $\mathbb{Q}(\theta)$ is ± 1, so it can be absorbed into A. Let $A = a + b\frac{\theta+1}{2}$, where $a, b \in \mathbb{Z}$. Then

$$\begin{aligned}\frac{3U^3 + \theta V^3}{2} &= \frac{3+\theta}{2}A^3\\ &= \frac{3+\theta}{2}(a + b\frac{1+\theta}{2})^3\\ &= \frac{3(a^3 - 18a^2b - 48ab^2 + 44b^3)}{2} + \frac{\theta(a^3 + 6a^2b - 24ab^2 - 28b^3)}{2}.\end{aligned}$$

Thus

$$U^3 = a^3 - 18a^2b - 48ab^2 + 44b^3, \quad V^3 = a^3 + 6a^2b - 24ab^2 - 28b^3. \tag{2.13}$$

If $3|U$, then $a \equiv b \mod 3$. Hence $a^3 \equiv b^3 \mod 9$. So $0 \equiv 3ab^2 \mod 9$, leading to $a \equiv b \equiv 0 \mod 9$, and hence $\gcd(U, V) > 1$, impossible. Therefore $3 \nmid U$. If $3|V$, then $a \equiv b \mod 3$, implying $3|U$, impossible. So $3 \nmid U, V$.

Let $L = \mathbb{Q}(\phi)$, where $\phi^3 - 12\phi - 10 = 0$. Then L has class number 3 and two fundamental units

$$\epsilon_1 = 1 + \phi, \quad \epsilon_2 = 3 + \phi, \quad \text{Norm}(\epsilon_1) = -1, \quad \text{Norm}(\epsilon_2) = 1.$$

Let $q_{13} = (13, \phi - 2)$ and $p_7 = (7, \phi)$. Then

$$(2) = p_2^3; \quad (3) = p_3^3; \quad (13) = p_{13}q_{13}^2,$$

where

$$\begin{aligned}(2 + \phi) &= p_2p_3,\\ (4 + \phi) &= p_2p_{13},\\ (-2 + \phi) &= p_2q_{13},\\ (-\phi^2 - 2\phi + 2) &= p_2^2p_{11},\\ (\phi^2 - 2\phi - 6) &= p_2^2p_7.\end{aligned}$$

We have

$$\phi \equiv 9 \mod p_{13}, \quad \phi \equiv 2 \mod q_{13},$$

and

$$\begin{aligned}U^3 =&(a+(-\phi^2-2\phi+2)b)(a^2+(\phi^2+2\phi-20)ab+(-6\phi^2+14\phi+32)b^2),\\ V^3 =&(a+(\phi^2-2\phi-6)b)(a^2+(-\phi^2+2\phi+12)ab+(-2\phi^2-2\phi+8)b^2).\end{aligned}$$

The gcd of $(a+(-\phi^2-2\phi+2)b)$ and $(a^2+(\phi^2+2\phi-20)ab+(-6\phi^2+14\phi+32)b^2)$ divides $78(2+\phi)$. The gcd of $(a+(\phi^2-2\phi-6)b)$ and $(a^2+(-\phi^2+2\phi+12)ab+(-2\phi^2-2\phi+8)b^2)$ divides $18(2-\phi)$.
Let

$$(a+(-\phi^2-2\phi+2)b) = p_2^{i_1}p_3^{i_2}p_{13}^{i_3}q_{13}^{i_4}\mathcal{X}^3,$$

where $\mathcal{X}$ is an ideal in $\mathcal{O}_L$. Taking norms gives

$$U^3 = 2^{i_1}3^{i_2}13^{i_3+i_4}X_1^3,$$

where $X_1 = \mathrm{Norm}(\mathcal{X})$. So

$$i_1 = i_2 = 0, \quad i_3+i_4 \equiv 0 \mod 3.$$

Thus

$$(a+(-\phi^2-2\phi+2)b) = \mathcal{X}^3,$$

or

$$(a+(-\phi^2-2\phi+2)b) = (13)\mathcal{X}^3,$$

or

$$(a+(-\phi^2-2\phi+2)b) = (2\phi^2-9\phi-3)\mathcal{X}^3.$$

The later two cases cannot occur. Otherwise, $a-6b \equiv 0 \mod 13$. Setting $a = 6b+13c$ gives

$$U^3 = 13^2(4b^4+12b^2c-13c^3), \quad V^3 = 13(20b^3+156b^2c+312bc^2+169c^3).$$

Then $13|U, V$, contradicting $\gcd(U,V) = 1$. Thus

$$\begin{aligned}(a+(-\phi^2-2\phi+2)b) =&\mathcal{X}^3,\\ (a^2+(\phi^2+2\phi-20)ab+(-6\phi^2+14\phi+32)b^2) =&\bar{\mathcal{X}}^3,\end{aligned} \tag{2.14}$$

where $\mathcal{X}\bar{\mathcal{X}} = (U)$.
Similarly

$$\begin{aligned}(a+(\phi^2-2\phi-6)b) =&\mathcal{Y}^3,\\ (a^2+(-\phi^2+2\phi+12)ab+(-2\phi^2-2\phi+8)b^2) =&\bar{\mathcal{Y}}^3,\end{aligned} \tag{2.15}$$

where $\mathcal{Y}\bar{\mathcal{Y}} = (V)$.
If $\mathcal{X} \sim 1$, then from (2.14)

$$\begin{aligned}a+(-\phi^2-2\phi+2)b =&\epsilon_1^{i_1}\epsilon_2^{i_2}X_1^3, \quad X_1 \in \mathcal{O}_L,\\ a^2+(\phi^2+2\phi-20)ab+(-6\phi^2+14\phi+32)b^2 =&\epsilon_1^{-i_1}\epsilon_2^{-i_2}\bar{X_1}^3, \quad X_1\bar{X_1} = U.\end{aligned} \tag{2.16}$$

If $\mathcal{X} \sim p_2$, then from (2.14)

$$\begin{aligned}a+(-\phi^2-2\phi+2)b =&\frac{1}{4}\epsilon_1^{i_1}\epsilon_2^{i_2}X_2^3, \quad X_2 \in \mathcal{O}_L,\\ a^2+(\phi^2+2\phi-20)ab+(-6\phi^2+14\phi+32)b^2 =&\frac{1}{2}\epsilon_1^{-i_1}\epsilon_2^{-i_2}\bar{X_2}^3, \quad X_2\bar{X_2} = 2U.\end{aligned} \tag{2.17}$$

If $\mathcal{X} \sim p_2^2$, then from (2.14)

$$\begin{aligned} a + (-\phi^2 - 2\phi + 2)b =& \frac{1}{2}\epsilon_1^{i_1}\epsilon_2^{i_2}X_3^3, \quad X_3 \in \mathcal{O}_L, \\ a^2 + (\phi^2 + 2\phi - 20)ab + (-6\phi^2 + 14\phi + 32)b^2 =& \frac{1}{4}\epsilon_1^{-i_1}\epsilon_2^{-i_2}\bar{X}_2^{\,3}, \quad X_3\bar{X}_3 = 2U. \end{aligned} \tag{2.18}$$

Similarly:
If $\mathcal{Y} \sim 1$, then from (2.15)

$$\begin{aligned} a + (\phi^2 - 2\phi - 6)b =& \epsilon_1^{j_1}\epsilon_2^{j_2}Y_1^3, \quad Y_1 \in \mathcal{O}_L, \\ a^2 + (-\phi^2 + 2\phi + 12)ab + (-2\phi^2 - 2\phi + 8)b^2 =& \epsilon_1^{-j_1}\epsilon_2^{-j_2}\bar{Y}_1^{\,3}, \quad Y_1\bar{Y}_1 = V. \end{aligned} \tag{2.19}$$

If $\mathcal{Y} \sim p_2$, then from (2.15)

$$\begin{aligned} a + (\phi^2 - 2\phi - 6)b =& \frac{1}{4}\epsilon_1^{j_1}\epsilon_2^{j_2}Y_2^3, \quad Y_2 \in \mathcal{O}_L, \\ a^2 + (-\phi^2 + 2\phi + 12)ab + (-2\phi^2 - 2\phi + 8)b^2 =& \frac{1}{2}\epsilon_1^{-j_1}\epsilon_2^{-j_2}\bar{Y}_2^{\,3}, \quad Y_2\bar{Y}_2 = 2V. \end{aligned} \tag{2.20}$$

If $\mathcal{Y} \sim p_2^2$, then from (2.15)

$$\begin{aligned} a + (\phi^2 - 2\phi - 6)b =& \frac{1}{2}\epsilon_1^{j_1}\epsilon_2^{j_2}Y_3^3, \quad Y_3 \in \mathcal{O}_L, \\ a^2 + (-\phi^2 + 2\phi + 12)ab + (-2\phi^2 - 2\phi + 8)b^2 =& \frac{1}{4}\epsilon_1^{-j_1}\epsilon_2^{-j_2}\bar{Y}_3^{\,3}, \quad Y_3\bar{Y}_3 = 2V. \end{aligned} \tag{2.21}$$

The equations (2.16) – (2.18) and (2.19) – (2.21) give the following equations respectively in $\mathcal{O}_L$:

$$\begin{aligned} a + (-\phi^2 - 2\phi + 2)b =& \frac{1}{\mu}\epsilon_1^{i_1}\epsilon_2^{i_2}X_i^3, \\ a^2 + (\phi^2 + 2\phi - 20)ab + (-6\phi^2 + 14\phi + 32)b^2 =& \frac{1}{\mu'}\epsilon_1^{-i_1}\epsilon_2^{-i_2}\bar{X}_i^{\,3}, \end{aligned}$$

where $(\mu, \mu') = (1,1), (4,2), (2,4)$; and

$$\begin{aligned} a + (\phi^2 - 2\phi - 6)b =& \frac{1}{\upsilon}\epsilon_1^{j_1}\epsilon_2^{j_2}Y_3^3, \quad Y_j \in \mathcal{O}_L, \\ a^2 + (-\phi^2 + 2\phi + 12)ab + (-2\phi^2 - 2\phi + 8)b^2 =& \frac{1}{\upsilon'}\epsilon_1^{-j_1}\epsilon_2^{-j_2}\bar{Y}_3^{\,3}, \quad Y_j\bar{Y}_j = V, \end{aligned}$$

where $(\upsilon, \upsilon') = (1,1), (4,2), (2,4)$.
We accordingly have equations in $\mathcal{O}_L$:
(2.22)

$$(a + (-\phi^2 - 2\phi + 2)b)(a^2 + (-\phi^2 + 2\phi + 12)ab + (-2\phi^2 - 2\phi + 8)b^2) = \frac{1}{\mu\upsilon'}\epsilon_1^r\epsilon_2^s X_i^3\bar{Y}_j^{\,3},$$

(2.23)

$$(a + (\phi^2 - 2\phi - 6)b)(a^2 + (\phi^2 + 2\phi - 20)ab + (-6\phi^2 + 14\phi + 32)b^2) = \frac{1}{\mu'\upsilon}\epsilon_1^{-r}\epsilon_2^{-s}\bar{X}_i^{\,3}Y_j^3,$$

where $r(= i_1 - j_1) = 0, \pm 1$, $s(= i_2 - j_2) = 0, \pm 1$.
Now $3 \nmid UV$, so $(X_i), (\bar{X}_i), (Y_j), (\bar{Y}_j)$ are coprime to p_3. Then for $\alpha \in \mathcal{O}_L$ and $p_3 \nmid (\alpha)$, we have $p_3|(\alpha^2 - 1)$. Therefore $3 = p_2^3|(\alpha^2 - 1)^3 \equiv \alpha^6 - 1 \mod 3$. Hence $\alpha^3 \equiv \pm 1$

mod 3. It follows that $X_i^3 \bar{Y_j}^3 \equiv \pm 1 \mod 3$. Since $\mu, \mu', \upsilon, \upsilon' \equiv \pm 1 \mod 3$, equation (2.22) gives

(2.24) $$(a+b)(a^2+ab+b^2)+b(a^2+ab+b^2)\phi^2 \equiv \pm \epsilon_1^r \epsilon_2^s \mod 3,$$

and equation (2.23) gives

(2.25) $$(a+b)(a^2-b^2)+b^2(a-b)\phi - b(a^2-b^2)\phi^2 \equiv \pm {\epsilon_1^{-r}} {\epsilon_2^{-s}} \mod 3.$$

We have

Table 1: Possible Values Of (r,s)

(r,s)	$\epsilon_1^r \epsilon_2^s$	${\epsilon_1^{-r}} {\epsilon_2^{-s}}$
(-1,-1)	$-\phi^2+2\phi+7$	$\phi^2+4\phi+3$
(-1,0)	$-\phi^2+\phi+11$	$\phi+1$
(-1,1)	$-2\phi^2+2\phi+23$	$-2\phi^2+6\phi+7$
(0,-1)	$\phi^2-3\phi-3$	$\phi+3$
(0,0)	1	1
(0,1)	$\phi+3$	$\phi^3-3\phi-3$
(1,-1)	$-2\phi^2+6\phi+7$	-2$\phi^2+2\phi+23$
(1,0)	$\phi+1$	$-\phi^2+\phi+11$
(1,1)	$\phi^2+4\phi+3$	$-\phi^2+2\phi+7$

Comparing coefficients of ϕ, equation (2.24) eliminates all but $(r,s) = (0,-1),(0,0),(1,-1)$, with corresponding units $\zeta = \epsilon_1^r \epsilon_2^s = \phi^2-3\phi-3, 1, -2\phi^2+6\phi+7$. It remains to treat the nine pairs of equations at (2.22), (2.23):

(2.26)

$$C_1\colon (a+(-\phi^2-2\phi+2)b)(a^2+(-\phi^2+2\phi+12)ab+(-2\phi^2-2\phi+8)b^2) = \frac{1}{\lambda}\cdot\zeta\cdot cube,$$

$$C_2\colon (a+(\phi^2-2\phi-6)b)(a^2+(\phi^2+2\phi-20)ab+(-6\phi^2+14\phi+32)b^2) = \frac{1}{\lambda'}\cdot\zeta\cdot cube,$$

where $(\lambda,\lambda') = (1,1),(4,2),(2,4)$ and $\zeta \in \{\phi^2-3\phi-3, 1, -2\phi^2+6\phi+7\}$.
For each pairs of equations in (2.26), the elliptic curve Chabauty routine in Magma [1] works on either C_1 or C_2. The result is recorded in the following table, where $\emptyset$ means there are no solutions.

Table 2: Solutions Corresponding to the Values Of (λ, r, s)

λ	(r,s)	Curve	Rank	Cubic model	(a,b)
1	(0,-1)	C_2	1	$y^2 = x^3+9(-17\phi^2+16\phi+193)$	$\emptyset$
1	(0,0)	C_2	1	$y^2 = x^3+(360802\phi^2-6430320\phi-7101783)$	$(\pm1,0)$
1	(-1,1)	C_1	0	$y^2 = x^3+(2168127\phi^2-6430320\phi-7101783)$	$\emptyset$
4	(0,-1)	C_1	0	$y^2 = x^3+(9204\phi^2-27144\phi-30732)$	$\emptyset$
4	(0,0)	C_1	0	$y^2 = x^3+(-312\phi^2+312\phi+4212)$	$\emptyset$
4	(1,-1)	C_1	1	$y^2 = x^3+(28\phi^2-68\phi-83)$	$\emptyset$
2	(0,-1)	C_2	1	$y^2 = x^3+(28\phi^2-68\phi-83)$	$\emptyset$
2	(0,0)	C_1	0	$y^2 = x^3+(64584\phi^2+247104\phi+169533)$	$\emptyset$

2	(1,-1)	C_2	1	$y^2 = x^3 + (7\phi^2 - 20\phi - 23)$	$\emptyset$

So $(a, b) = (\pm 1, 0)$. Hence $|U| = |V| = 1$. Thus $X = 2$ and $(x, y) = (\pm 2, \pm 5)$. $\square$

3. Equation $y^2 = x^6 - 47$

In this section, we will prove Theorem 1.2.

Proof. Equation $y^2 = x^6 - 47$ is equivalent to

$$Y^2 = X^6 - 47Z^6,$$

where X, Y, Z are coprime. We have

$$(X^3 - Y)(X^3 + Y) = 47Z^6.$$

The $\gcd(X^3 - Y, X^3 + Y)$ divides $\gcd(2X^3, 2Y)$, so divides 2. We can choose the sign of Y such that $47|X^3 + Y$.
Case gcd is 1:

$$X^3 + Y = 47V^6, \quad X^3 - Y = U^6, \quad \gcd(U, V) = 1.$$

So

$$2X^3 = 47V^6 + U^6, \quad \gcd(U, V) = 1.$$

If $13 \nmid UV$, then $2X^3 \equiv \pm 1 \pm 47 \mod 13$. Thus $4X^6 \equiv (1 \pm 5)^2 \mod 13$. So $\pm 4 \equiv \pm 3 \mod 13$, impossible. Therefore $13|UV$. If $13|U$, then $2X^3 \equiv 47V^6 \equiv \pm 5 \mod 13$. Thus $4X^6 \equiv 25 \equiv -1 \mod 13$. So $\pm 4 \equiv -1 \mod 13$, impossible. If $13|V$, then $2X^3 \equiv U^6 \mod 13$. Thus $4X^6 \equiv U^{12} \equiv 1 \mod 13$. So $\pm 4 \equiv \pm 1 \mod 13$, impossible.
Case gcd is 2:
Then

$$X^3 + Y = 47 \cdot 2 \cdot V^6, \quad X^3 - Y = 2^5 \cdot U^6, \quad \gcd(U, V) = 1,$$

or

$$X^3 + Y = 47 \cdot 2^5 \cdot V^6, \quad X^3 - Y = 2 \cdot U^6, \quad \gcd(U, V) = 1;$$

So

$$X^3 = 47V^6 + 16U^6, \quad \gcd(U, V) = 1,$$

or

$$X^3 = 47 \cdot 2^4 \cdot V^6 + U^6, \quad \gcd(U, V) = 1.$$

The latter case gives $(X/V^2)^3 = 752 + (U^3/V^3)^2$. The elliptic curve $y^2 = x^3 - 752$ has rank 0, and the trivial torsion subgroup, implying $V = 0$. So we only need to consider the case

$$X^3 = 16U^6 + 47V^6. \tag{3.1}$$

From $63^3 = 16 \cdot 5^3 + 47$, we would like to show that $X = 63$, $|U| = |V| = 1$.
If $3|U$, then from (3.1), we have $X^3 \equiv 47V^6 \equiv 2 \mod 9$. Thus $X^6 \equiv 4 \mod 9$, so $1 \equiv 4 \mod 9$, impossible. So $3 \nmid U$. If $3|V$, then $X^3 \equiv 16U^6 \equiv -2 \mod 9$. Thus $X^6 \equiv 4 \mod 9$, impossible. So $3 \nmid V$. Therefore $X^3 \equiv 0 \mod 9$, giving $3|X$.
From (3.1), we also have $2 \nmid X$, V.
Let $K = \mathbb{Q}(\theta)$, where $\theta = \sqrt{-47}$. K has the class number 5, the trivial fundamental

unit group, and the ring of integers $\mathcal{O}_K = \mathbb{Z}[\frac{1+\theta}{2}]$. The class group of K is generated by the ideal $I = (2, \frac{1+\theta}{2})$. Now

$$(X)^3 = (4U^3 + \theta V^3)(4U^3 - \theta V^3). \tag{3.2}$$

Let J be a common ideal dividing both factors on the right side. Then

$$J|(8U^3), \quad J|(2\theta V^3), \quad J^2|(X)^3.$$

Taking norms gives

$$\mathrm{Norm}(J)|64U^6, \quad \mathrm{Norm}(J)|4 \cdot 47 \cdot V^6, \quad \mathrm{Norm}(J)|X^3.$$

But $2 \nmid X$, so $\mathrm{Norm}(J)|\gcd(X^3, U^6, 47V^6) = 1$. Therefore $(4U^3 + \theta V^3)$ and $(4U^3 - \theta V^3)$ are coprime ideals. Thus

$$(4U^3 + \theta V^3) = \mathcal{A}^3,$$

where $\mathcal{A}$ is an ideal in $\mathcal{O}_K$. K has class number 5 with the trivial unit group, hence

$$4U^3 + \theta V^3 = A^3 \tag{3.3}$$

with $A \in \mathcal{O}_K$. Let $A = u + v\frac{(1+\theta)}{2}$, where $u, v \in \mathbb{Z}$. Then

$$A^3 = (3/2u^2v + 3/2uv^2 - 11/2v^3)\theta + u^3 + 3/2u^2v - 69/2uv^2 - 35/2v^3.$$

$A^3 \in \mathbb{Z}[\theta]$ implies $u^3 + 3/2u^2v - 69/2uv^2 - 35/2v^3 \in \mathbb{Z}$, hence $\frac{u^2v - uv^2 - v^3}{2} \in \mathbb{Z}$. If $2 \nmid v$, then $\frac{u^2-u-1}{2} \in \mathbb{Z}$, impossible. So $2|v$. Therefore $A \in \mathbb{Z}[\theta]$. Let

$$4U^3 + \theta V^3 = (a + b\theta)^3,$$

where $a, b \in \mathbb{Z}$. Taking norms gives

$$X = a^2 + 47b^2.$$

$2|X$ implies $2 \nmid a, b$; $3|X$ implies $3 \nmid a, b$. Expanding $(a + b\theta)^3$ gives

$$\begin{aligned} 4U^3 &= a(a^2 - 141b^2), \\ V^3 &= b(3a^2 - 47b^2). \end{aligned} \tag{3.4}$$

In the second equation, we have

$$\gcd(b, 3a^2 - 47b^2) = \gcd(b, 3a^2) = \gcd(b, 3) = 1.$$

Further, V is odd so b is odd. $3a^2 - 47b^2 | V^3$ so $3a^2 - 47b^2$ is odd, hence a is even. Thus $a^2 - 141b^2$ is odd, so $4|a$. If $47|a$, then $47|v^3$ and $47|U^3$. So $47|\gcd(U, V)$, contradicting $\gcd(U, V) = 1$. Hence $47 \nmid a$, so $\gcd(a, a^2 - 141b^2) = 1$. Therefore from (3.4), we have

$$a = 4A^3, \quad b = B^3, \quad 3a^2 - 47b^2 = C^3, \quad a^2 - 141b^2 = D^3,$$

where $A, B, C, D \in \mathbb{Z}$, $AD = U$, $CB = V$.
Because $\gcd(U, V) = \gcd(a, b) = \gcd(a, 141) = \gcd(b, 3) = 1$, we have A, B, C, D are coprime. Further, $3, 47 \nmid a$, so $3, 47 \nmid A, D$; $2, 3 \nmid b$ so $2, 3 \nmid B, C$. Now

$$48A^6 - 47B^6 = C^3,$$

$$16A^6 - 141B^6 = D^3.$$

We will show $|A| = |B| = 1$ and $C = 1$, $D = -5$. Indeed, we have

$$3C^3 - D^3 = 128A^6,$$

$$C^3 - 3D^3 = 376B^6.$$

Note that $C^3 \equiv 3D^3 \mod 8$ and $2 \nmid C$, so

$$C \equiv 3D \mod 8.$$

Also $C^3 \equiv 3D^3 \mod 47$ and $47 \nmid D$, so

$$D \equiv -5C \mod 47.$$

Let $L = \mathbb{Q}(\phi)$, where $\phi = \sqrt[3]{3}$. L has class number 1, the ring of integers $\mathcal{O}_L = \mathbb{Z}[\phi]$, and a fundamental unit $\epsilon = \phi^2 - 2$ of norm 1. The ideal $(2) = p_2 q_2$, where $p_2 = (-1+\phi)$ and $q_2 = (1 + \phi + \phi^2)$. The ideal $(47) = p_{47} q_{47}$, where $p_{47} = (2 + \phi + 2\phi^2)$ and $q_{47} = (2 - 10\phi + 3\phi^2)$. Now

$$(C - D\phi)(C^2 + CD\phi + D^2\phi^2) = 2^3 \cdot 47 \cdot B^6.$$

Because

$$\gcd(C - D\phi, C^2 + CD\phi + D^2\phi^2) = \gcd(C - D\phi, 3D^2\phi^2) = \gcd(C - D\phi, \phi^5) = 1,$$

the two factors on the left are coprime.
We note that

$$C - D\phi \equiv C(1 + 5\phi) \equiv 0 \mod p_{47},$$
$$C - D\phi \equiv D(3 - \phi) \equiv 0 \mod p_2^3.$$

Thus

$$C - D\phi = (-1)^h \epsilon^i p_2^j p_{47}^k G^6,$$

where $G \in \mathcal{O}_L$, and $0 \le h \le 1$, $0 \le i, j, k \le 5$. Taking norms gives

$$2^3 \cdot 47 \cdot B^6 = (-1)^h 2^j 47^k \mathrm{Norm}(G)^6.$$

So h is even, $j \equiv 3 \mod 6$, $k \equiv 1 \mod 6$. Thus $(h, j, k) = (0, 3, 1)$. Then

$$C - D\phi = \epsilon^i (13 - 10\phi + \phi^2) G^6.$$

We claim that $i = 5$.
If $i \equiv 0 \mod 2$, then

$$C - D\phi = (13 - 10\phi + \phi^2)(M + N\phi + P\phi^2)^2, \quad M, N, P \in \mathbb{Z}.$$

Comparing coefficients of ϕ^2 gives

$$M^2 - 20MN + 13N^2 + 26MP + 6NP - 30P^2 = 0,$$

which is locally unsolvable at 2. Thus i is odd.
If $i = 3$, then

$$C - D\phi = (13 - 10\phi + \phi^2)(M + N\phi + P\phi^2)^3, \quad M, N, P \in \mathbb{Z}.$$

Comparing coefficients of ϕ^2 gives

$$M^3 - 30M^2N + 39MN^2 + 3N^3 + 39M^2P + 18MNP - 90N^2P - 90MP^2 + 117NP^2 + 9P^3 = 0,$$

which is locally unsolvable at 3.
If $i = 1$, then

$$C - D\phi = (-56 + 23\phi + 11\phi^2)(M + N\phi + P\phi^2)^3, \quad M, N, P \in \mathbb{Z}.$$

Comparing coefficients of ϕ^2 gives

$$11M^3 + 69M^2N - 168MN^2 + 33N^3 - 168M^2P + 198MNP + 207N^2P + 207MP^2 - 504NP^2 + 99P^3 = 0,$$

which is locally unsolvable at 3. Therefore $i = 5$, equivalently, on taking $i = -1$, we have

$$C - D\phi = (1 + 5\phi)G^6.$$

It follows that
$$(C - D\phi)(3C^3 - D^3) = 2(1 + 5\phi)(2AG)^6,$$
or
$$2(1 + 5\phi)(x - \phi)(3x^3 - 1) = y^2,$$
where $x = \frac{C}{D}$ and $y = 2(1 + 5\phi)(2AG)^3/D^2$, representing an elliptic curve over L. The cubic model is
$$y^2 = x^3+(-30\phi^2+174\phi+36)x^2+(9012\phi^2+5040\phi-12708)x+(207576\phi^2-409536\phi+449064).$$
This curve has rank 2. The Chabauty routine in Magma [1] shows $\frac{C}{D} = \frac{-1}{5}$. Hence $C = 1$, $D = -5$, and $|A| = |B| = 1$. Therefore the only solutions to $y^2 = x^6 - 47$ are $x = \pm\frac{63}{10}$ and $y = \pm\frac{249953}{10^3}$.

$\square$

References

[1] W. Bosma, J. Cannon, and C. Playoust. *The Magma algebra system. I. The user language.* J. Symbolic Comput. 24(3-4): 235-265, 1997.

[2] A. Bremner, and N. Tzanakis, *On the equation* $Y^2 = X^6 + k$. Ann. Sci. Math. Quebec 35 (2011), no. 2, 153-174.

[3] E. V. Flynn and J. L. Wetherell, *Finding rational points on bielliptic genus 2 curves,* Manuscripta Math. 100:4 (1999), 519-533.

[4] S. Siksek and M. Stoll, *Partial descent on hyperelliptic curves and the generalized Fermat equation* $x^3 + y^4 + z^5 = 0$, Bull. London Math. Soc. 44:1 (2012), 151-166.

[5] J. H. Silverman, *The arithmetic of elliptic curves, 2nd edition.* Graduate Texts in Mathematics 106, Springer(2009).

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