Cyclic Sieving Phenomenon on Matchings

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January 11, 2018
Overview

1. Definitions

2. Examples / Theorems

3. Main result: matchings on \([2n]\)

4. Future problems
 definitions

- $X$ is a finite set.
- $C = \langle c \rangle$ is a cyclic group of order $n$ acting on $X$.
- $X(q)$ is a polynomial in $q$ with nonnegative integer coefficients.

**Definition [Reiner-Stanton-White]**

The triple $(X, X(q), C)$ exhibits the cyclic sieving phenomenon (CSP) if for all integers $d$, the number of elements fixed by $c^d$ equals the evaluation $X(\zeta^d)$ where $\zeta$ is a $n$-th root of unity.
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Elements of $X$ in a $C$-orbit have the same stabilizer subgroup. We call the cardinality of the stabilizer subgroup stabilizer-order for the orbit.

**Definition [Reiner-Stanton-White]**

The coefficient $\alpha_\ell$ defined uniquely by the expansion

$$X(q) \equiv \sum_{\ell=0}^{n-1} \alpha_\ell q^\ell \mod q^n - 1$$

has the following interpretation: $\alpha_\ell$ counts the number of $C$-orbits on $X$ for which the stabilizer-order divides $\ell$. 
A (complete) matching on $[2n] := \{1, 2, \ldots, 2n\}$ is a partition of $[2n]$ of type $(2, 2, \ldots, 2)$.

Q: What is the number of matchings on $[2n]$?
A: $\frac{(2n)!}{n! \cdot 2^n} = 1 \cdot 3 \cdot 5 \cdots (2n - 1)$.

For a matching $\{(i_1, j_1), \ldots, (i_n, j_n)\}$ where $i_r < j_r$ for all $r$, two blocks $(i_r, j_r)$ and $(i_s, j_s)$ form a crossing if $i_r < i_s < j_r < j_s$.

Q: What is the number of non-crossing matchings on $[2n]$?
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Example 1: Non-crossing matchings

Let $NCM_3$ be the set of non-crossing matchings on [6].
Let $C_6 = \langle (1\ 2\ 3\ 4\ 5\ 6) \rangle$ and let $X(q) = \frac{1}{[4]_q} [6]_{3_q} = 1 + q^2 + q^3 + q^4 + q^6$.

For $\zeta = e^{\frac{2\pi i}{6}} = \frac{1}{2} + \frac{\sqrt{3}i}{2}$,

$X(\zeta^0) = 5$, $X(\zeta^2) = X(\zeta^4) = 2$ and $X(\zeta^3) = 3$ and $X(\zeta) = X(\zeta^5) = 0$.

The triple $(NCM_3, X(q), C_6)$ exhibits the CSP.
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The triple $(NCM_3, X(q), C_6)$ exhibits the CSP.
Theorem: non-crossing matchings

- Let \([n]_q = 1 + q + \cdots + q^{n-1}\) denote the \(q\)-analog of \(n\).
- Let \([n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q\).
- Let \(\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}\) denote the \(q\)-binomial coefficient.

Theorem [Sagan]

Let \(N\!C\!M_n\) be the set of non-crossing matchings on \([2n]\). Let \(C_{2n} = \langle (1 \ 2 \ \ldots \ 2n) \rangle\). Let \(\text{Cat}(n)\) denote the \(q\)-Catalan polynomial,

\[
\text{Cat}(n) = \frac{1}{[n+1]_q \binom{2n}{n}_q}.
\]

Then the triple \((N\!C\!M_n, \text{Cat}(n), C_{2n})\) exhibits the CSP.
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Theorem [Sagan]

Let $NCM_n$ be the set of non-crossing matchings on $[2n]$. Let $C_{2n} = \langle(1\ 2\ \ldots\ 2n)\rangle$. Let $Cat(n)$ denote the $q$-Catalan polynomial,

$$Cat(n) = \frac{1}{[n+1]_q}\binom{2n}{n}_q.$$

Then the triple $(NCM_n, Cat(n), C_{2n})$ exhibits the CSP.
Example 2

Let $OCM_3$ be the set of one-crossing matchings on $[6]$. Let $X(q) = \binom{6}{1}q = 1 + q + q^2 + q^3 + q^4 + q^5$. For $\zeta = e^{\frac{2\pi i}{6}} = \frac{1}{2} + \frac{\sqrt{3}i}{2}$,

$X(\zeta^0) = 6$ and $X(\zeta^1) = X(\zeta^2) = X(\zeta^3) = X(\zeta^4) = X(\zeta^5) = 0$. 
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Theorem: $k$-crossing matchings for $k \leq 3$

Theorem [Liang-Bowling]

Let $OCM_n$ be the set of one-crossing matchings on $[2n]$. Then the triple $(OCM_n, \lceil \frac{2n}{n-2} \rceil q, C_{2n})$ exhibits the CSP.

Similar results proved by Liang-Bowling:

- on two-crossing matchings with $\frac{[n+3]q}{[2]q} \lceil \frac{2n}{n-3} \rceil q$,
- on three-crossing matchings with $\frac{1}{[3]q} \lceil \frac{n+5}{2} \rceil q \lceil \frac{2n}{n-4} \rceil q + \lceil \frac{2n}{n-3} \rceil q$. 
Q: What about the whole set $P_n$ of matchings on $[2n]$ rather than a set of matchings of a particular number of crossings?

Q: Does $q$-analog of $\frac{(2n)!}{n! \cdot 2^n}$ or $1 \cdot 3 \cdot 5 \cdots (2n - 1)$ work?

A: No, it does not even when $n = 2$.

The $q$-analog $\frac{[4]_q!}{[2]_q![2]_q^2}$ is not a polynomial.

The $q$-analog $[1]_q[3]_q = 1 + q + q^2$. 
Example: the set $P_2$ of matchings on $[4]$

The triple $(P_2, q^2 + 2, C_4)$ exhibits the CSP.

Note that $q^2 + 2 = (q^2 + 1) + 1 = \text{Cat}(2) + \binom{4}{0}q$. 

\[
\{(1, 2), (3, 4)\} \quad \{(1, 3), (2, 4)\} \quad \{(1, 4), (2, 3)\}
\]
Example: the set $P_3$ of matchings on $[6]$

Elements in $P_3$ (courtesy of Thomas Lam, used with permission)

By Liang-Bowling, the CSP polynomial is $X(q) = \text{Cat}(3) + \binom{6}{1}q + \frac{6q}{2q} + 1$. 
Example: $P_4$ and $P_5$ and $P_6$

1. There are $1 \cdot 3 \cdot 5 \cdot 7 = 105$ elements in $P_4$.
2. There are $1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 = 945$ elements in $P_5$.
3. There are $1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 = 10395$ elements in $P_6$. 
Proposition

- $X$ is any finite set.
- $C = \langle c \rangle$ is a cyclic group of order $N$ acting on $X$.
- For $d \mid N$, let $b_d$ be the number of $C$-orbits on $X$ of order $N/d$.

Proposition [K.]

Let $f(q)$ be

$$f(q) = \sum_{d \mid N} b_d (q^{N-d} + q^{N-2d} + \cdots + q^d + 1).$$

Then, the triple $(X, f(q), C)$ exhibits the CSP.

In our situation, we need to set $N = 2n$. 
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Proposition

- For \( d|2n \), let \( a_{d,n} \) be the number of elements in \( P_n \) fixed by \( 2\pi/d \) rotation, or equivalently fixed by \( c^{2n/d} \).

Proposition [K.]
The coefficients \( b_d \) satisfy

\[
a_{d,n} = \sum_{d|r} \frac{2n}{r} b_r.
\]
Let $d | 2n$. Then the sequence $a_{d,n}$ satisfies the following recurrence relations depending on the parity of $d$.

1. If $d$ is even, then $a_{d,n} = a_{d,n-\frac{d}{2}} + (2n - d)a_{d,n-d}$ with the initial condition $a_{d,\frac{d}{2}} = 1$ and $a_{d,d} = d + 1$.

2. If $d$ is odd, then $a_{d,n} = (2n - d)a_{d,n-d}$ with the initial condition $a_{d,d} = d$. 
Proof idea: recursion for $a_{d,n}$ for even $d$

Suppose $d = 12$ and $n = 18$. Then $a_{d,n} = a_{12,18}$ counts the number of matchings on $[2n] = [36]$ fixed by $2\pi/d = \pi/6$ rotation.

\[
a_{12,18} = a_{12,12} + (36 - 12)a_{12,6} + (2n - d)a_{d,n-d}
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partner of $2n$ is $n$

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If $d$ is even, then

$$a_{d,n} = 1 + n \sum_{i \geq 0} \frac{(2i + 1)!}{(i + 1)!} \left( \frac{2n}{d} - 1 \right) \left( \frac{d}{2} \right)^i.$$ 

If $d$ odd, then

$$a_{d,n} = \prod_{i=1}^{n/d} (2i - 1)d.$$
Example: $n = 6$

\[ X_6(q) = b_1(q^{11} + \cdots + q + 1) + b_2(q^{10} + \cdots + q^2 + 1) \\
+ b_3(q^9 + \cdots + 1) + b_4(q^8 + q^4 + 1) + b_6(q^6 + 1) + b_{12} \]

\[
\begin{bmatrix}
12 & 6 & 4 & 3 & 2 & 1 \\
0 & 6 & 0 & 3 & 2 & 1 \\
0 & 0 & 4 & 0 & 2 & 1 \\
0 & 0 & 0 & 3 & 0 & 1 \\
0 & 0 & 0 & 2 & 1 \\
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\end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_6 \\ b_{12} \end{bmatrix} = \begin{bmatrix} a_{1,6} \\ a_{2,6} \\ a_{3,6} \\ a_{4,6} \\ a_{6,6} \\ a_{12,6} \end{bmatrix} = \begin{bmatrix} 10395 \\ 331 \\ 27 \\ 13 \\ 7 \\ 1 \end{bmatrix}
\]

\[ X_6(q) = 837(q^{11} + \cdots + q + 1) + 52(q^{10} + \cdots + q^2 + 1) \\
+ 5(q^9 + \cdots + 1) + 4(q^8 + q^4 + 1) + 3(q^6 + 1) + 1 \\
= 837q^{11} + 889q^{10} + \cdots + 837q + 902. \]
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b_2 \\
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0 & 0 & 0 & 0 & 0 & 1
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\begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
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= 
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\]
Future problems

1. For $3 < k < \binom{n}{2}$ find polynomials $X_{n,k}(q)$ such that the set of $k$-crossing matchings together with $X_{n,k}(q)$ and $C_{2n}$ exhibits the CSP.

2. Investigate homomesy phenomenon on $P_n$. In other words, for a group action on $P_n$, not necessarily cyclic shifts, find a statistic such that the average value of the statistic on every orbit is the same as the average of the statistic over $P_n$. 
Thank You!