Cyclic Sieving Phenomenon on Matchings

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Abstract. Reiner-Stanton-White defined the cyclic sieving phenomenon (CSP) associated to a finite cyclic group action on a finite set and a polynomial. Sagan observed the CSP on the set of non-crossing matchings on \([2n] := \{1, 2, \ldots, 2n\}\) using the cyclic group \(C_{2n}\) generated by a cyclic shift of order \(2n\) and the \(q\)-Catalan polynomial

\[
X(q) = \frac{1}{[n+1]_q} [2n]_q.
\]

Bowling-Liang presented a similar result on the set of one-crossing matchings with a completely different proof. We focus on the set \(P_n\) of all matchings on \([2n]\) rather than a set of matchings of a particular number of crossings. We find the number of elements in \(P_n\) fixed by \(c^{2n/d}\) rotations, or \(c^{2n/d}\) where \(c = (1 2 \ldots 2n)\), for \(d|2n\). We find the polynomials \(X_n(q)\) such that \(P_n\) together with \(X_n(q)\) and \(C_{2n}\) exhibits the CSP.

Keywords: cyclic sieving phenomenon, matchings

1 Introduction

The cyclic sieving phenomenon was defined by Reiner-Stanton-White [3]. Let \(X\) be a finite set. Let \(C\) be a cyclic group generated by an element \(c \in C\) of order \(n\) acting on \(X\). Let \(X(q)\) be a polynomial with integer coefficients in a variable \(q\).

Definition 1 ([3]). The triple \((X, X(q), C)\) exhibits the cyclic sieving phenomenon (CSP) if for all integers \(d\), the number of elements of \(X\) fixed by \(c^d\) equals the evaluation \(X(\zeta^d)\) where \(\zeta = e^{2\pi i/n}\) is a \(n\)th-root of unity.

In particular, since \(X(1) = |X|\), the polynomial \(X(q)\) can be thought of as a generating function for the set \(X\). Reiner-Stanton-White ([3], Theorem 1.1) proved that \((X, X(q), C_n)\) exhibits the CSP with the collection \(X\) of all \(k\)-elements subsets of \([n] := \{1, 2, \ldots, n\}\), the cyclic group \(C_n = \langle c \rangle\) where \(c = (1 2 \ldots n)\), and \(q\)-binomial coefficients

\[
X(q) = \binom{n}{k}_q := \frac{[n]!_q}{[k]!_q [n-k]!_q}
\]

where \([m]!_q := [m]_q[m-1]_q \ldots [2]_q[1]_q\) and \([m]_q := 1 + q + q^2 + \cdots + q^{m-1}\).

Example 2. Take \(n = 4\) and \(k = 2\), and let \(C_4 = \langle c \rangle\) where \(c = (1 2 3 4)\) and let \(X(q) = \binom{3}{2}_q = 1 + q + 2q^2 + q^3 + q^4\). For \(\zeta = e^{2\pi i} = i\), we see that \(X(\zeta^0) = 6\), and it means...
all 2-subsets are fixed by the identity. We also check that \( X(\zeta^2) = 2 \), and we interpret two 2-subsets, namely \( \{1, 3\} \), \( \{2, 4\} \), are fixed by \( c^2 = (1 \ 3)(2 \ 4) \), and \( X(\zeta) = X(\zeta^3) = 0 \) which means no 2-subset is fixed by \( c = (1 \ 2 \ 3 \ 4) \) or \( c^3 = (1 \ 4 \ 3 \ 2) \).

Let \( \tau \) be a matching on \([2n]\). If two pairs \((a, b)\) and \((c, d)\) in \(\tau\) satisfy \(a < c < b < d\), we say the pairs create a crossing. For example, in Figure 1 there are two noncrossing matchings and one one-crossing matching in \(P_2\). The CSP is observed on the set of noncrossing matchings on \([2n]\).

**Theorem 3** ([5], Theorem 8.1). Let \( X \) be the collection of noncrossing matchings on \([2n]\). Let \( C_{2n} = \langle c \rangle \) where \( c = (1 \ 2 \ldots \ 2n) \). Let \( X(q) \) be the \(q\)-Catalan number,

\[
X(q) = \frac{1}{[n+1]_q} \left[ \begin{array}{c} 2n \\ n \end{array} \right]_q.
\]

Then the triple \((X, X(q), C_{2n})\) exhibits the CSP.

This result is found in a survey of CSP by Sagan [5]. This is a special case, when \(m = 2\), of Theorem 4 in Rhoades’ paper [4].

**Theorem 4** ([4], Theorem 1.3). Let \( \lambda = (n, n, \ldots, n) \vdash mn \) be a rectangular partition and let \( X = SYT(\lambda) \) be the set of standard Young tableaux of shape \( \lambda \). Let \( C = \mathbb{Z}/mn\mathbb{Z} \) act on \( X \) by jeu-de-taquin promotion. Then the triple \((X, C, X(q))\) exhibits the CSP, where \( X(q) \) is the \(q\)-analogue of the hook length formula

\[
X(q) = f^\lambda(q) := \frac{[mn]!_q}{\prod_{(i,j) \in \lambda} [h_{ij}]_q}.
\]

As stated in Petersen-Pylyavskyy-Rhoades [2], Dennis White set up a bijection between standard Young tableaux in \( SYT(n, n) \) and noncrossing matchings, which has not published. For \( T \in SYT(n, n) \), form a corresponding sequence of parentheses by placing a left parenthesis under each number in the first row and a right parenthesis under each number in the second row. Then match the parentheses. For example,

\[
T = \begin{array}{ccc}
1 & 3 & 4 \\
2 & 5 & 6
\end{array} \quad \mapsto \quad \begin{array}{cccc}
1 & 2 & 3 & 4 & 5 & 6 \\
( & ( & ( & ) & ) & )
\end{array} \quad \mapsto \quad \{(1,2), (3,6), (4,5)\}.
\]

We see that the hook length formula is reduced to the \(q\)-Catalan number when \( \lambda = (n, n) \). Thus the bijection by Dennis White and Theorem 4 directly implies Theorem 3.

Motivated by the CSP on the non-crossing matchings, Bowling-Liang [1] showed that the CSP is observed on the set of one-crossing matchings in \(P_n\).
Theorem 5 ([1], Theorem 1). Let $X$ be the collection of one-crossings matchings on $[2n]$. Let $C_{2n} = \langle c \rangle$ where $c = (1 2 \ldots 2n)$. Let

$$X(q) = \left[ \frac{2n}{n-2} \right]_q.$$ 

Then the triple $(X, X(q), C_{2n})$ exhibits the CSP.

Bowling-Liang first noticed that the number of one-crossing matchings on $[2n]$ is $\binom{2n}{n-2}$. Then they set $X(q) := \left[ \frac{2n}{n-2} \right]_q$ to be the $q$-analogue of $\binom{2n}{n-2}$, and they enumerated the number of one-crossing matchings fixed by $c^d$. They finished their proof by showing that the evaluation $X(\zeta_d^d)$ is the number of one-crossing matchings fixed by $c^d$.

Both Theorem 3 and Theorem 5 established the CSP on the set of matchings of a particular number of crossings. One could keep working on the set of elements of a certain number of crossings, but we change our focus on the whole set $P_n$ of matchings on $[2n]$. Our main result, Theorem 17, is that we find polynomials $X_n(q)$ for $X = P_n$ and $C_{2n} = \langle c \rangle$ where $c = (1 2 \ldots 2n)$ such that the triple $(P_n, X_n(q), C_{2n})$ exhibits the CSP.

The rest of this abstract is structured as follows. We first find the polynomials $X_n(q)$ when $n$ is any prime in Section 2. In Section 3, we find the number of matchings in $P_n$ fixed by $\frac{2\pi}{d}$ rotations, or $c^{2n/d}$ where $c = (1 2 \ldots 2n)$, for $d | 2n$. Then we find the polynomials $X_n(q)$ for any $n \geq 1$. Section 2 is a special case of Section 3, but we do keep the section because it would be helpful to understand Section 3.

2 The case of a prime

We first examine when $n = 2$.

Example 6. Let $X_2(q) = q^2 + 2$. Let $C_4 = \langle c \rangle$ where $c = (1 2 3 4)$. Then the triple $(P_2, X_2(q), C_4)$ exhibits the CSP. Observe that there is one element in $P_2$ fixed by $c$ and $c^3$.

![Figure 1: The set $P_2$ of matchings on $[4]$](image-url)
Thus if we let $\zeta = e^{2\pi i/4} = i$ be a fourth-root of unity, then we see that $X_2(1) = X_2(-1) = 3$ and $X_2(i) = X_2(-i) = 1$, and this shows the triple $(P_2, X_2(q), C_4)$ exhibits the CSP.

Wire diagrams on $2n$ points on a circle are a good visualization of matchings on $[2n]$. Let $d$ divide $2n$, then the action of $c^{2n/d}$ on matchings can be interpreted as the $2\pi/d$ rotation of corresponding wire diagrams. We state and prove following lemmas.

**Lemma 7.** Define a sequence $t_n (n \geq 1)$ by $t_n = t_{n-1} + (2n-2)t_{n-2}$ and $t_1 = 1$, $t_2 = 3$. Then the number of elements in $P_n$ fixed by $c^n = (1 2 \ldots 2n)^n = (1 n+1)(2 n+2) \ldots (n 2n)$, the 180 degree rotation, is $t_n$.

**Proof.** by Proposition 14, we have $t_n = a_{2,n}$ which satisfies $a_{2,n} = a_{2,n-1} + (2n-2)a_{2,n-2}$ with $a_{2,1} = 1$ and $a_{2,2} = 3$.

**Remark 8.** Lemma 7 is true for not only primes but also for any natural numbers, but the following lemma is only true for odd primes.

**Remark 9.** The sequence $t_n$ is A047974 in the OEIS [6].

**Lemma 10.** For a prime $n \geq 3$, the number of elements in $P_n$ fixed by $c^d$ for $d \in \{1, 3, 5, \ldots, 2n-1\}\{n\}$ is 1. The number of elements in $P_n$ fixed by $c^d$ for $d \in \{2, 4, 6, \ldots, 2n-2\}$ is $n$.

The following lemma will be used to prove that coefficients of $X_n(q)$ are integers in Theorem 12.

**Lemma 11.** Let $\alpha_n = \frac{1}{2n} \left( \frac{(2n)!}{2^n n!} - t_n - n + 1 \right)$ and $\beta_n = \frac{t_n - 1}{n}$. For any prime $n \geq 3$, both $\alpha_n$ and $\beta_n$ are integers.

**Proof.** By (3.3) in Proposition 15, we have $t_n = a_{2,n} = 1 + n \sum_{i \geq 0} \frac{(2i+1)!}{(i+1)!} \frac{n-1}{(2i+1)}$. Then we see that $\beta_n = (t_n - 1)/n = \sum_{i \geq 0} \frac{(2i+1)!}{(i+1)!} \frac{n-1}{(2i+1)}$ is an integer. Notice that $\frac{(2n)!}{2^n n!} = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$ is the product of first $n$ odd numbers, and it is divisible by $n$. Thus we see that $\frac{(2n)!}{2^n n!} - n$ is divisible by $2n$. We also observe that $t_n - 1 = n \sum_{i \geq 0} \frac{(2i+1)!}{(i+1)!} \frac{n-1}{(2i+1)}$ is divisible by $2n$ because $(n-1)/(2i+1)$ is even for all $i \geq 0$. Thus $\alpha_n$ is an integer.

In the following theorem, we present the polynomials $X_n(q)$ for any odd prime $n$.

**Theorem 12.** Let $n \geq 3$ be a prime number. Let $C_{2n} = \langle c \rangle$ where $c = (1 2 \ldots 2n)$. Let $X_n(q)$ be the polynomial

$$X_n(q) = \alpha_n \cdot \frac{q^{2n} - 1}{q - 1} + \beta_n \cdot \frac{q^{2n} - 1}{q^2 - 1} + \frac{n - 1}{2} \cdot \frac{q^{2n} - 1}{q^n - 1} + 1$$

where $\alpha_n = \frac{1}{2n} \left( \frac{(2n)!}{2^n n!} - t_n - n + 1 \right)$ and $\beta_n = \frac{t_n - 1}{n}$. Then, the triple $(P_n, X_n(q), C_{2n})$ exhibits the CSP.
Proof. By Lemma 11, the polynomial $X_n(q)$ is with integral coefficients. Let $\zeta = e^{2\pi i/2n}$ be a $(2n)$-th root of unity. By Lemma 7 and Lemma 10, we need to check:

$$
X_n(\zeta^d) = 1 \quad \text{for } d \in \{1, 3, 5, \ldots, 2n-1\}\setminus\{n\}
$$

$$
X_n(\zeta^d) = n \quad \text{for } d \in \{2, 4, 6, \ldots, 2n-2\}
$$

$$
X_n(\zeta^0) = X_n(1) = |P_n| = \frac{(2n)!}{2^n \cdot n!}
$$

$$
X_n(\zeta^n) = X_n(-1) = t_n.
$$

For $d \in \{1, 3, 5, \ldots, 2n-1\}\setminus\{n\}$, since $\zeta^d \neq 1$ and $(\zeta^d)^2 \neq 1$ and $(\zeta^d)^n = -1$, thus we have $X_n(\zeta^d) = 1$. For $d \in \{2, 4, 6, \ldots, 2n-2\}$, since $\zeta^d \neq 1$ and $(\zeta^d)^2 \neq 1$ and $(\zeta^d)^n = 1$, thus we have $X_n(\zeta^d) = n$. We compute that

$$
X_n(1) = a_n \cdot 2n + \frac{t_n - 1}{n} \cdot n + \frac{n - 1}{2} \cdot 2 + 1 = \frac{(2n)!}{2^n \cdot n!}
$$

$$
X_n(-1) = \frac{t_n - 1}{n} \cdot n + 1 = t_n
$$

as desired. \qed

Example 13. Let $n = 3$. By Theorem 12,

$$
X_3(q) = \frac{q^6 - 1}{q - 1} + 2\frac{q^6 - 1}{q^2 - 1} + \frac{q^6 - 1}{q^3 - 1} + 1 = (q^5 + q^4 + q^3 + q^2 + q + 1) + 2(q^4 + q^2 + 1) + (q^3 + 1) + 1
$$

since $t_3 = t_2 + 4t_1 = 7$ and $a_3 = \frac{1}{6}(15 - t_3 - 3 + 1) = 1$. Thus, we have $X_3(q) = q^5 + 3q^4 + 2q^3 + 3q^2 + q + 5$.

3 The general case

In this section, for all $n \geq 1$ we want to find the polynomials $X_n(q)$ for which the triple $(P_n, X_n(q), C_{2n})$ exhibits the CSP. To do this we find the number of elements in $P_n$ fixed by $C_{2n/d}$.

3.1 Matchings fixed by $C^k$

Suppose $d$ divides $2n$. First, we want to count the number of elements in $P_n$ fixed by $2\pi/d$ rotation. Let $a_{d,n}$ be the number of such elements in $P_n$. For example, Lemma 7 tells us that $a_{2,n} = a_{2,n-1} + (2n-2)a_{2,n-2}$ with $a_{2,1} = 1$ and $a_{2,2} = 3$. 


**Proposition 14.** Suppose $2n = dk$ for some $k \in \mathbb{Z}$. Then the sequence $a_{d,n}$ satisfies the following recurrence relations depending on the parity of $d$.

1. If $2$ divides $d$, then
   \[ a_{d,n} = a_{d,n - \frac{d}{2}} + (2n - d)a_{d,n - d} \]  
   with the initial condition $a_{d, \frac{d}{2}} = 1$ and $a_{d,d} = d + 1$.

2. If $d$ is not divisible by $2$, then
   \[ a_{d,n} = (2n - d)a_{d,n - d} \]  
   with the initial condition $a_{d,d} = d$.

**Proof sketch.** We count the number of ways to construct $\tau \in P_n$ which is fixed by $2\pi/d$ rotation. We break up into cases: $\tau(2n) = n$ or $\tau(2n) \neq n$. Suppose $2 \mid d$. If $\tau(2n) = n$ then the number of ways to construct $\tau$ is $a_{d,n - \frac{d}{2}}$. If $\tau(2n) \neq n$ then we have $(2n - d)a_{d,n - d}$ ways to construct $\tau$.

Now suppose $2 \nmid d$. If $\tau(2n) = n$ then there are $a_{d,n - d}$ ways to construct $\tau$. If $\tau(2n) \neq n$ then there are $(2n - d - 1) \cdot a_{d,n - d}$ ways to construct $\tau$. 
From the recurrence relations, we find formulas for \( a_{d,n} \).

**Proposition 15.** If 2 divides \( d \), then

\[
a_{d,n} = 1 + n \sum_{i \geq 0} (2i + 1)! \left( \frac{2n}{d} - 1 \right) \left( \frac{d}{2} \right)^i.
\]

(3.3)

If \( d \) is not divisible by 2, then

\[
a_{d,n} = \frac{n}{d} \prod_{i=1}^{\frac{n}{d}} (2i - 1)d.
\]

(3.4)

If \( \gcd(2n,l) = \gcd(2n,m) \), then \( \langle c^l \rangle = \langle c^m \rangle \). From this observation, we have the following proposition.

**Proposition 16.** Let \( 1 \leq l < m \leq 2n \) such that \( \gcd(2n,l) = \gcd(2n,m) \). Then, a matching \( \tau \in P_n \) is fixed by \( c^l \) if and only if \( \tau \) is fixed by \( c^m \). Therefore, if the triple \((P_n, X_n(q), C_{2n})\) exhibits the CSP then \( X_n(\zeta^l) = X_n(\zeta^m) \).

### 3.2 The polynomial \( X_n(q) \)

Here is our main result.

**Theorem 17.** Let \( C_{2n} = \langle c \rangle \) where \( c = (1 \ 2 \ \ldots \ 2n) \). Let \( X_n(q) \) be

\[
X_n(q) = \sum_{d | 2n} b_{d,n} \frac{q^{2n} - 1}{q^d - 1} = \sum_{d | 2n} b_{d,n} (q^{2n-d} + q^{2n-2d} + \ldots + q^{2d} + q^d + 1)
\]

(3.5)

where the coefficients \( b_{d,n} \) satisfy

\[
a_{d,n} = \sum_{d | r} \frac{2n}{r} b_{r,n}.
\]

(3.6)

Then, the triple \((P_n, X_n(q), C_{2n})\) exhibits the CSP.

We break down the proof of this theorem into two parts. In the first part, we prove \( X_n(\zeta^k) \) is equal to the number of elements fixed by \( c^k \). In the second part, we prove the coefficients \( b_{d,n} \) are integers.

**Proof of Theorem 17 part 1.** Let \( d \) divide \( 2n \), and \( 2n = dk \). Observe that the number of elements in \( P_n \) fixed by \( c^k \) is \( a_{d,n} \), the number of elements fixed by \( 2\pi/d \) rotation. We need to show that the evaluation \( X_n(\zeta^k) = a_{d,n} \). Note that

\[
\left. \frac{q^{2n} - 1}{q^r - 1} \right|_{q = \zeta^k} = 0
\]
unless $2n|kr$. If $2n|kr$ or equivalently $d|r$, we have

$$b_{r,n} \frac{q^{2n} - 1}{q^r - 1} \bigg|_{q=\zeta^k} = b_{r,n}(q^{2n-r} + q^{2n-2r} + \cdots + q^{2r} + q^r) \bigg|_{q=\zeta^k} = b_{r,n}(1 + 1 + \cdots + 1) = b_{r,n} \frac{2n}{r}.$$ 

Let $j \in [2n]$ such that $j \nmid 2n$ and $\gcd(2n,j) = k$. By Proposition 16 the number of elements in $P_n$ fixed by $c^j$ is equal to the number of elements fixed by $c^k$. So we need to that the evaluation $X_n(\zeta^j) = X_n(\zeta^k) = a_{d,n}$. Note that

$$\frac{q^{2n} - 1}{q^r - 1} \bigg|_{q=\zeta^j} = 0$$

unless $2n|jr$. Suppose $2n|jr$, then $dk|jr$ which forces $d|r$ because $\gcd(2n,j) = k$. We see that

$$b_{r,n} \frac{q^{2n} - 1}{q^r - 1} \bigg|_{q=\zeta^j} = b_{r,n}(q^{2n-r} + q^{2n-2r} + \cdots + q^{2r} + q^r) \bigg|_{q=\zeta^j} = b_{r,n}(1 + 1 + \cdots + 1) = b_{r,n} \frac{2n}{r},$$

and thus $X_n(\zeta^k) = a_{d,n}$. \hfill \Box

The following lemma shows how to express the coefficients $b_{d,n}$ in terms of $a_{r,n}$’s in (3.5).

**Lemma 18.** Let $P = \{p_1, p_2, \ldots, p_m\}$ be the set of primes that divide $2n/d$. Then

$$\frac{2n}{d} b_{d,n} = \sum_{S \subseteq P} (-1)^{|S|} a_{\prod_{p \in S} p, n} \tag{3.7}$$

$$= a_{d,n} - (a_{dp_1,n} + \cdots + a_{dp_m,n}) + \cdots + (-1)^m a_{dp_1p_2\ldots p_m,n}.$$  

Proof. Let $D_{2n}$ be the poset on the set of all divisors of $2n$ partially ordered by $i \leq j$ if and only if $j$ is divisible by $i$. Let $g(d) = a_{d,n}$ and $f(d) = \frac{2n}{d} b_{d,n}$. The equation (3.6) can be written as $g(d) = \sum_{r|d \leq r} f(r)$. By the dual form of M"obius inversion formula (Proposition 3.7.2 in [7]), we have $f(d) = \sum_{d \leq r} \mu(d,r) g(r)$. Since

$$\mu(d,r) = \begin{cases} (-1)^t & \text{if } r/d \text{ is a product of } t \text{ distinct primes} \\ 0 & \text{otherwise} \end{cases}$$

(see Example 3.8.4 in [7]) and the proof follows. \hfill \Box

To complete the proof of Theorem 17, we need the following lemmas.
Lemma 19. Let $m$ be an odd integer. If $n \geq 2$ then the number $\prod_{i=1}^{2n} (2i-1) - 1 = 1 \cdot 3 \cdot \ldots \cdot (2a+1)m - 1 \cdot (2n+1)m - 1$ is divisible by $2n+1$. If $n = 1$ then the number $\prod_{i=1}^{2m} (2i-1) - 1$ is even but not divisible by 4.

Lemma 20. Let $2n = 2^ro_1 \cdot \ldots \cdot p_k^\ell$ for some odd primes $p_1, \ldots, p_k$. Let $d$ be any odd divisor of $2n$. Then, $a_{d,n} = a_{2d,n}$ is divisible by $2^ro_1$.

Now we complete the proof of Theorem 17.

Proof sketch of Theorem 17 part 2. We prove the coefficients $b_{d,n}$ are integers. Suppose $d = 2n$ then $b_{2n,n} = 1$ which is divisible by $n$. Suppose $d$ is an even number less than $2n$. Suppose $p_1, \ldots, p_m$ be primes that divide $2n/d$. By (3.7) and (3.3), we see that

$$\frac{2n}{d} b_{d,n} = a_{d,n} - (a_{dp_1,n} + \ldots + a_{dp_m,n}) + \ldots + (-1)^m a_{dp_1 \ldots p_m,n}$$

$$= n \sum_{i \geq 0} \frac{(2i+1)!}{(i+1)!} \left( \frac{2n}{2i+1} \right) \left( \frac{d}{2} \right)^i + \ldots + (-1)^m \left( \frac{2n}{dp_1 \ldots p_m} - 1 \right) \left( \frac{dp_1 \ldots p_m}{2} \right)^i$$

Thus, we have $b_{d,n} = \frac{d}{2} \sum_{i \geq 0} \frac{(2i+1)!}{(i+1)!} \left( \frac{2n}{2i+1} \right) \left( \frac{d}{2} \right)^i + \ldots + (-1)^m \left( \frac{2n}{dp_1 \ldots p_m} - 1 \right) \left( \frac{dp_1 \ldots p_m}{2} \right)^i$ which is an integer. Now suppose $d$ is odd. Let $\frac{2n}{d} = 2^ro_1 \ldots p_{t,i}^{\ell}$. We observe that

$$\frac{2n}{d} b_{d,n} = 2^ro_1 \ldots p_{t,i}^{\ell} b_{d,n}$$

where $A_{odd} = a_{d,n} - (a_{dp_1,n} + \ldots + a_{dp_{t,i},n}) + \ldots + (-1)^{t} a_{dp_1 \ldots p_{t,i},n}$ and $A_{even} = a_{2d,n} - (a_{2dp_1,n} + \ldots, a_{2dp_{t,i},n}) + \ldots + (-1)^{t} a_{2dp_1 \ldots p_{t,i},n}$. By Lemma 20 $A_{odd} - A_{even}$ is divisible by $2^ro$. We observe that

$$A_{even} = n \sum_{i \geq 0} \frac{(2i+1)!}{(i+1)!} \left( \frac{2n}{2i+1} \right) \left( \frac{d}{2} \right)^i + \ldots + (-1)^{t} \left( \frac{2n}{2dp_1 \ldots p_{t,i}} - 1 \right) \left( \frac{2dp_1 \ldots p_{t,i}}{2} \right)^i$$

which is divisible by $n$, and thus is divisible by $p_{t,i}^{\ell}$. We claim that each term in $A_{odd}$ is divisible by $p_{t,i}^{\ell}$. Let $a_{y,n}$ be any term in $A_{odd}$, say $y = dp_{i_1} \ldots p_{i_j}$. If $p_{i} \in \{p_{i_1}, \ldots, p_{i_j}\}$ then we see that

$$a_{y,n} = \prod_{i=1}^{n/y} (2i - 1) y = \prod_{i=1}^{n/y} (2i - 1)$$

which is divisible by $p_{t,i}^{\ell}$. Since $n/y \geq p_{t,i}^{\ell-1} \geq t_i$, we have $a_{y,n}$ is divisible by $p_{t,i}^{\ell}$. If $p_{i} \notin \{p_{i_1}, \ldots, p_{i_j}\}$ then $a_{y,n}$ is divisible by

$$\prod_{i=1}^{2^ro_1 p_{t,i}^{\ell-1} \ldots p_{t,i}^{\ell-1}} (2i - 1)$$
which is divisible by $p_1^{t_1}$ because $p_1^{t_1} \in \{2i - 1 : 1 \leq i \leq 2^{r_0-1}p_1^{t_1}p_2^{t_2-1} \ldots p_{\ell}^{t_{\ell}-1}\}$. Thus our claim is proved. For the same reason, $a_{y,n}$ is divisible by each $p_i^{t_i}$ for $1 \leq i \leq \ell$. Therefore, we conclude that $A_{\text{odd}} - A_{\text{even}}$ is divisible by $\frac{2^n}{d}$, and hence all coefficients $b_{d,n}$'s in $X_n(q)$ are integers.

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\section*{References}


