The cd-indices of intervals in the uncrossing partial order on matchings

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Abstract

We study flag enumeration in intervals in the uncrossing partial order on matchings. We produce a recursion for the cd-indices of intervals in the uncrossing poset $P_n$. We explicitly describe the matchings by constructing an order-reversing bijection. We obtain a recursion for the ab-indices of intervals in the poset $\hat{P}_n$, the poset $P_n$ with a unique minimum $\hat{0}$ adjoined.

Keywords: partially ordered set, flag enumeration, cd-index, uncrossing partial order, matching, affine permutation, Bruhat order

1 Introduction

A graded poset $P$ with a unique maximum and a unique minimum is Eulerian if, for every non-trivial interval $[x, y]$ in $P$, the number of elements of odd rank in $[x, y]$ is equal to the number of elements of even rank in $[x, y]$. One nice property of Eulerian posets is their very simple Möbius functions $\mu_P(s, t) = (-1)^{\ell(s, t)}$. Another nice property of Eulerian posets is that they have a cd-index. A cd-index is a non-commutative generating functions, which is an efficient way to encode the flag enumeration of Eulerian posets. The cd-index arose from the work of Bayer-Billera [2] on flag $f$-vectors and flag $h$-vectors of Eulerian posets. Bayer-Klapper [3] and Stanley [14] proved the existence of the cd-indices of Eulerian posets. Ehrenborg-Readdy [7] showed the way to obtain the cd-index of some operations, for example, the cd-index of pyramid of $P$, the product of a poset $P$ with a chain of length one. Reading [13] presented a recursive formula for the cd-indices of intervals in the Bruhat order on a Coxeter group.

The uncrossing partially ordered set $P_n$ on matchings is defined on the set of matchings on $2n$ points on a circle represented with wires, with an order relation: $\tau' \leq \tau$ in $P_n$ if and
only if $\tau'$ is obtained by resolving a crossing of $\tau$. For example, if $\tau = \{(1,3),(2,4)\}$, $\tau' = \{(1,2),(3,4)\}$ and $\tau'' = \{(1,4),(2,3)\}$ in $P_2$ then $\tau' \leq \tau$ and $\tau'' \leq \tau$. This partial order has been studied by Alman-Lian-Tran [1], Huang-Wen-Xie [8], Kenyon [9], and Lam [11], [12]. The uncrossing posets emerged from studies of circular planar electrical networks. Circular planar networks are finite weighted undirected graphs embedded into a disk, with boundary vertices and interior vertices. By Curtis-Ingerman-Morrow [6] and de Verdière-Gitler-Vertigan [5], the electrical networks can be encoded with response matrices. By Lam [11] the space of response matrices for electrical networks has a cell structure, and this cell structure can be described by the uncrossing partial orders (see [11], Proposition 4.10). Let $\hat{P}_n$ denote $P_n$ with a unique minimum element $\hat{0}$ adjoined. The poset $\hat{P}_n$ was conjectured to be Eulerian by Alman-Lian-Tran [1], Huang-Wen-Xie [8], and Kim-Lee [10]. This conjecture was proved by Lam [12]. Thus it is natural for us to pay attention to the cd-index of the posets $\hat{P}_n$ and their intervals.

The main result in this work is a recursive formula in Theorem 18 for the cd-indices of intervals in the uncrossing poset $P_n$. To find this, we construct an order-reversing bijection in Theorem 10 between $P_n$ and $\mathcal{MP}_n$, an induced subposet of affine permutations $\tilde{S}_{2n}$ of type $(0,2n)$. We explicitly describe the elements in the induced subposet. Lam [11] introduced a representation of matchings in $P_n$: to a matching $\tau \in P_n$, associate $g_\tau : \mathbb{Z} \to \mathbb{Z}$ by

$$g_\tau(i) = \begin{cases} 
\tau(i) & \text{if } i < \tau(i) \\
\tau(i) + 2n & \text{if } i > \tau(i). 
\end{cases} \quad (1)$$

Lam showed that this map $\tau \mapsto g_\tau$ identifies $P_n$ with an induced subposet of dual Bruhat order of affine permutations of type $(n,2n)$ (see [11], Theorem 4.16). However, we need a map to affine permutations of type $(0,2n)$ in order to apply a technique in Reading [13] for finding recursions for the cd-indices of intervals in Bruhat order on Coxeter groups. We slightly modify Lam’s map $\tau \mapsto g_\tau$ to adapt Reading’s ideas to our situation. Finally, using the cd-indices of intervals in $P_n$, we present a recursion in Theorem 29 for the ab-indices of intervals in the poset $\hat{P}_n$.

The paper is structured as follows. In Section 2, we give basic definitions needed to understand this paper: posets, the cd-index, affine permutations, and Bruhat order. The definitions and notations in this section are mainly obtained from Stanley [15] and Björner-Brenti [4].

## 2 Preliminaries

In this section we give background information on partially ordered sets, the cd-index, affine permutations, and Bruhat order. The definitions and notations in this section are mainly obtained from Stanley [15] and Björner-Brenti [4].
2.1 Posets

Let $P$ be a poset and $x, y \in P$. If $x$ covers $y$ we write $x \triangleright y$. By an *induced subposet*, or *subposet* for short, of $P$, we mean a subset $Q$ of $P$ and a partial ordering on $Q$ such that for $x, y \in Q$ we have $x \leq y$ in $Q$ if and only if $x \leq y$ in $P$. For $x \leq y$ in $P$, a (closed) *interval* $[x, y] = \{z \in P : x \leq z \leq y\}$ is a subposet of $P$. Given two posets $P$ and $Q$, form their product $P \times Q$ on the set $\{(x, y) : x \in P, y \in Q\}$ such that $(x_1, y_1) \leq (x_2, y_2)$ in $P \times Q$ if $x_1 \leq x_2$ in $P$ and $y_1 \leq y_2$ in $Q$. A graded poset $P$ with a unique maximum and a unique minimum is *Eulerian* if, for every interval $[x, y]$ in $P$ where $x < y$, the number of elements of odd rank in $[x, y]$ is equal to the number of elements of even rank in $[x, y]$.

2.2 The cd-index

The cd-index arose from the work of Bayer-Billera [2] on flag $f$-vectors of Eulerian posets. They extended the $f$-vector and the $h$-vector of a polytope $\Delta$ to the flag $f$-vector and the flag $h$-vector of the order complex $\Delta(P)$ of a finite graded poset $P$.

Let $\Delta$ be a finite $(d - 1)$-dimensional simplicial complex with $f_i$ $i$-dimensional faces. The vector $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ is called the *$f$-vector* of $\Delta$. The vector $h(\Delta) = (h_0, h_1, \ldots, h_d)$, called the *$h$-vector*, is defined by the relation

$$
\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}
$$

where $f_{-1} = 1$ unless $\Delta = \emptyset$ [2].

Let $P$ be a graded poset, rank $n$, with a $\hat{0}$ and a $\hat{1}$. The *order complex* $\Delta(P)$ of $P$ is defined as follows: the vertices of $\Delta(P)$ are the elements of $P - \{\hat{0}, \hat{1}\}$, and the faces of $\Delta(P)$ are the chains of $P - \{\hat{0}, \hat{1}\}$. The order complex $\Delta(P)$ is a simplicial complex [2], and Bayer-Billera enumerated faces of this order complex $\Delta(P)$ as follows. For a chain $C$ in $P - \{\hat{0}, \hat{1}\}$, define $\rho(C) = \{\rho(x) : x \in C\}$. Let $[n] := \{1, 2, \ldots, n\}$ and let $2^{[n]}$ be the set of all subsets of $[n]$. For any $S \subseteq [n-1]$, define

$$
\alpha_P(S) = |\{C \subseteq P : C \text{ is a chain such that } \rho(C) = S\}|.
$$

We call the function $\alpha_P : 2^{[n-1]} \to \mathbb{N}$ the *flag $f$-vector*. We define

$$
\beta_P(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_P(T).
$$

The function $\beta_P : 2^{[n-1]} \to \mathbb{N}$ is called the *flag $h$-vector* of $P$. Then we can check for the order complex $\Delta = \Delta(P)$

$$

f_i(\Delta) = \sum_{|S|=i+1} \alpha_P(S)
$$

$$

h_i(\Delta) = \sum_{|S|=i} \beta_P(S).
$$

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Thus the flag $f$-vector $\alpha_P$ of the order complex $\Delta(P)$ counts flags (or chains) of $P - \{0, 1\}$ by length, which are faces of $\Delta(P)$ by dimension, as the $f$-vector counts faces of a polytope by dimension. The flag $h$-vector $\beta_P$ plays a role as the $h$-vector of a polytope.

Now we define ab-index of a poset and cd-index of an Eulerian poset. Define the \textit{characteristic monomial} $u_S$ of $S \subseteq [n - 1]$ by $u_S = e_1 e_2 \cdots e_{n-1}$, where

$$e_i = \begin{cases} 
    b & \text{if } i \in S \\
    a & \text{if } i \notin S.
\end{cases}$$

Define a noncommutative polynomial $\Psi_P(a, b)$, called the \textit{ab-index} of $P$, by

$$\Psi_P(a, b) = \sum_{S \subseteq [n-1]} \beta_P(S) u_S.$$ 

Thus $\Psi_P(a, b)$ is a noncommutative generating function for the flag $h$-vector $\beta_P$. By Bayer and Klapper’s theorem [3], if $P$ is an Eulerian poset of rank $n$, then there exists a polynomial $\Phi_P(c, d)$ in the noncommutative variables $c$ and $d$ such that $\Psi_P(a, b) = \Phi_P(a + b, ab + ba)$. The polynomial $\Phi_P(c, d)$ is called the \textit{cd-index} of $P$.

2.3 Affine permutations and Bruhat order

Let $\tilde{S}^*_n, n \geq 2$ be the group of all bijections $w$ of $\mathbb{Z}$ in itself such that

1. $w(x + n) = w(x) + n$ for all $x \in \mathbb{Z}$ and
2. $\sum_{x=1}^{n} w(x) - (1 + 2 + \cdots + n) = nk,$

with abbreviation $\tilde{S}^0_n$ by $\tilde{S}_n$. The \textit{window notation} of $w \in \tilde{S}_n$ is $w = [a_1, \ldots, a_n]$ if $w(i) = a_i$ for $i \in [n]$. As a set of generators for $\tilde{S}_n$ we take the set of periodic adjacent transpositions $\tilde{S} = \{s_0, s_1, \ldots, s_{n-1}\}$ where

$$s_i := \{1, 2, \ldots, i-1, i+1, i, i+2, \ldots, n\} \text{ for } 1 \leq i \leq n-1,$$

$$s_0 := \{0, 2, 3, \ldots, n-1, n+1\}.$$ Then, the pair $(\tilde{S}_n, S)$ is a Coxeter system and $\tilde{S}_n$ is the Coxeter group of \textit{affine permutations} of the integers.

There are several partial orders defined on $\tilde{S}_n$. We need Bruhat order and from now on, $\tilde{S}_n$ denotes the afine permutations, together with this order. For $w \in \tilde{S}_n$, a decomposition $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ with letters in $S$ is called a reduced decomposition for $w$ if $k$ is minimal. The word $i_1 i_2 \cdots i_k$ is called a reduced word for $w$. We say $k$ the length of $w$ and denote $\ell(w)$.

Fix a reduced decomposition for $w = t_1 t_2 \cdots t_k$ where $t_i \in S$ for all $1 \leq i \leq k$. The Bruhat order is defined by $v \leq_B w$ if and only if there is a reduced subword $t_{i_1} t_{i_2} \cdots t_{i_j}$ of $t_1 t_2 \cdots t_k$ for $v$ such that $1 \leq i_1 < i_2 < \cdots < i_j \leq k$. We will write $v \leq w$ for $v \leq_B w$ if there is no possibility of confusion.
3 The Uncrossing Posets

Let $P_n$ be the set of matchings on $[2n]$. Label $1, 2, \ldots, 2n$ in cyclic order on a circle. A lensless medial graph is a medial graph such that: (1) every wire begins and ends on the circle; (2) any two wires intersect at most once; and (3) no wire has a self intersection. Then, each $\tau \in P_n$ can be represented by a lensless medial graph $M(\tau)$. Now resolve any crossing in $M(\tau)$ in either of two ways:

This gives a new lensless medial graph $M' = M(\tau')$ for some matching $\tau'$ on $[2n]$. Then we define the partial order $\tau' \leq \tau$ on $P_n$ if the lensless medial graph $M(\tau')$ is obtained by resolving a crossing of the lensless medial graph $M(\tau)$. Let $c(\tau)$ be the number of crossings of a lensless medial graph for $\tau$. The poset $P_n$ is graded of rank $\binom{n}{2}$ with rank function given by $c(\tau)$. The poset $P_n$ has a unique maximum element, namely $\{(1,n+1), (2,n+2), \ldots, (n,2n)\}$, and Catalan number $\frac{1}{n+1}\binom{2n}{n}$ of minimal elements. Figure 1 is the Hasse diagram of $P_3$.

This partial order has been studied by Alman-Lian-Tran [1], Huang-Wen-Xie [8], Kenyon [9], and Lam [11], [12]. The uncrossing posets emerged from studies of circular planar electrical networks, finite weighted undirected graphs embedded into a disk, with boundary vertices and interior vertices. By Curtis-Ingerman-Morrow [6] and de Verdière-Gitler-Vertigan [5], the electrical networks can be encoded with response matrices. By Lam [11] the space of response matrices for electrical networks has a cell structure, and this cell structure can be described by the uncrossing partial orders (see [11], Proposition 4.10).

Let $\hat{P}_n$ denote $P_n$ with a unique minimum element $\hat{0}$ adjoined, where we let $c(\hat{0}) = -1$. The poset $\hat{P}_n$ was conjectured to be Eulerian by Alman-Lian-Tran [1], Huang-Wen-Xie [8], and Kim-Lee [10], and was proved to be so by Lam [12].

**Theorem 1** ([12], Theorem 1). $\hat{P}_n$ is an Eulerian poset.

Lam [12] used the map $\tau \mapsto g_\tau$ ([11], see (1) in Section 1) as the main tool for proving Theorem 1. Using this map, Lam showed that the number of odd elements equals the number of even rank elements in intervals in $\hat{P}_n$. To be specific, he first showed that the number of odd rank elements equals the number of even rank elements in any interval $[\tau, \eta] \subset P_n$ by decending induction on $c(\tau) + c(\eta)$. Then he showed the number of odd rank elements equals the number of even rank elements in any interval $[\hat{0}, \eta] \subset \hat{P}_n$ by establishing an involution $\sigma \mapsto s_i \cdot \sigma$ on the set $\{\sigma \in (\hat{0}, \eta) | s_i \cdot \sigma \neq \sigma\}$. 

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4 Modular Palindromic Permutations

Lam [11] mapped matchings $\tau \in P_n$ to affine permutations $g_\tau$ of type $(n, 2n)$. See (1) in Section 1. Lam showed that this map $\tau \mapsto g_\tau$ identifies $P_n$ with an induced subposet of dual Bruhat order of affine permutations of type $(n, 2n)$ (Theorem 4.16 in [11]), through Theorem 8.3.7 in Björner-Brenti [4] which characterizes affine Bruhat order in terms of a matrix which tracks inversions.

We notice that if we slightly modify Lam’s map it is possible to show that $P_n$ is identified with an induced subposet of dual Bruhat order of affine permutations $\tilde{S}_{2n}$ of type $(0, 2n)$ without Theorem 8.3.7 in [4]. We believe that this identification is necessary to apply a technique in Reading [13] for finding recursions for the cd-indices of intervals in Bruhat order on Coxeter groups. Moreover, we are able to explicitly describe the elements in the induced subposet of affine permutations as modular palindromic permutations (see Theorem 10).

For these reasons, we slightly modify the map $\tau \mapsto g_\tau$ to define an injective map $\phi : P_n \rightarrow \tilde{S}_{2n}$ as follows. For a matching $\tau \in P_n$, define $h_\tau : [2n] \rightarrow \mathbb{Z}$ by

$$h_\tau(i) = \begin{cases} 
\tau(i) - n & \text{if } i < \tau(i) \\
\tau(i) + n & \text{if } i > \tau(i).
\end{cases}$$
Now define \( \phi : P_n \to \tilde{S}_{2n} \) by \( \phi(\tau) = [h_\tau(1), h_\tau(2), \ldots, h_\tau(2n)] \) in the window notation of the affine permutation group \( \tilde{S}_{2n} \).

**Example 2.** Let \( \tau = \{(1,4), (2,6), (3,5)\} \) and \( \tau' = \{(1,3), (2,6), (4,5)\} \) in \( P_3 \). The images of \( \tau \) and \( \tau' \) under Lam’s map are \( g_\tau = [g_\tau(i)]_{i=1}^6 = [4,6,5,7,9,8] \) and \( g_{\tau'} = [g_{\tau'}(i)]_{i=1}^6 = [3,6,7,5,10,8] \). On the other hands, the images under the map \( \phi \) are \( \phi(\tau) = [1,3,2,4,6,5] = s_2s_5 \in \tilde{S}_6 \) and \( \phi(\tau') = [0,3,4,2,7,5] = s_2s_0s_3s_5 \in \tilde{S}_6 \).

With the map \( \phi \), we explicitly describe the subposet of \( \tilde{S}_{2n} \) which is isomorphic to the dual of \( P_n \). First, we define a modular palindromic permutation and a subset \( MP_n \) of the affine permutation group \( \tilde{S}_{2n} \).

**Definition 3.** An affine permutation \( w \in \tilde{S}_{2n} \) is modular palindromic if \( w \) has a reduced decomposition \( w = s_{i_1}s_{i_2} \ldots s_{i_{2k}} \) with \( |i_{2k+1-r} - i_r| = n \) for all \( r \in [k] \).

**Example 4.** Let \( \tau = \{(1,4), (2,6), (3,5)\} \) and \( \tau' = \{(1,3), (2,6), (4,5)\} \) in \( P_3 \) as in Example 2. Both \( \phi(\tau) = s_2s_5 \) and \( \phi(\tau') = s_2s_0s_3s_5 \) are modular palindromic. We also observe that \( \phi \) reverses the order, in other words, \( \tau' \leq \tau \) and \( \phi(\tau') \geq \phi(\tau) \).

\[
\begin{array}{c}
6 & 1 & 5 \\
5 & 6 & 1 \\
4 & 3 & 2
\end{array}
\quad \iff \quad
\begin{array}{c}
6 & 1 & 5 \\
5 & 6 & 1 \\
4 & 3 & 2
\end{array}
\quad \phi(\tau') = s_2s_0s_3s_5 \geq s_2s_5 = \phi(\tau)

**Example 5.** Let \( n = 3 \). Let \( w = s_0s_5s_1s_4s_2s_3 \in \tilde{S}_6 \). Notice that \( w \) is a modular palindromic permutation. The window notation of the permutation is \( w = [2,3,7,0,4,5] \). Assume \( w = \phi(\tau) \) for some \( \tau \in P_3 \). Since \( w(3) = 7 \), we must have \( h_\tau(3) = 7 \), which impossible because \( 1 \leq h_\tau(3) \leq 5 \). Thus \( w \not\in \phi(P_3) \).

The previous example shows that not all modular palindromic permutations are images of matchings under the map \( \phi \). Which conditions are needed for modular palindromic permutations to be images of matchings under the map \( \phi \)? We need the following definition.

**Definition 6.** The subset \( MP_n \) of \( \tilde{S}_{2n} \) is the set of all modular palindromic permutations \( w \in \tilde{S}_{2n} \) such that no reduced word of \( w \) contains a subword of \( n \) consecutive integers.

**Remark 7.** The consecutive integers need not be in adjacent positions in the reduced word.

Now we use \( MP_n \) as a poset, equipped with Bruhat order.

**Example 8.** Let \( n = 2 \). We see that \( MP_2 \) contains \( e, s_0s_2 \) and \( s_1s_3 \). Notice that all length four modular palindromic permutations, whose reduced words are 1023, 0132, 2130 and 3201 respectively, have a subword of 2-consecutive integers, and thus \( MP_2 = \{e, s_0s_2, s_1s_3\} \) and the followings are the Hasse diagrams of \( P_2 \) and the dual of \( MP_2 \), respectively.
Example 9 (revisited). Let $n = 3$. Let $w = s_0s_5s_1s_4s_2s_3 \in \tilde{S}_6$ with a reduced word $051423$. Notice that $w$ is a modular palindromic permutation, which has a subword $543$, a 3 consecutive integers. Thus $w \notin MP_3$.

The following theorem is one of the main results in this paper.

**Theorem 10.** The map $\phi$ is an order-reversing bijection between $P_n$ and $MP_n$.

To prove this theorem, we first state and prove a lemma.

**Lemma 11.** The image of $P_n$ under the map $\phi$ is contained in $MP_n$. In other words, $\phi(P_n) \subseteq MP_n$.

**Proof.** First, we prove that $\phi(\tau)$ is modular palindromic using decreasing induction on the ranks of $\tau \in P_n$.

(i) Base case: the unique maximum element $\hat{1} = \{(1, n + 1), (2, n + 2), \ldots, (n, 2n)\}$. Notice that $\phi(\hat{1}) = e \in MP_n$ as required.

(ii) Induction step: suppose $\tau \leq \hat{1}$ and $\phi(\tau) \in MP_n$. Choose a crossing generated by a pair of wires $(a, \tau(a))$ and $(b, \tau(b))$ such that $a < b < \tau(a) < \tau(b)$. We can resolve the crossing in two ways.

Case 1: from $\{(a, \tau(a)), (b, \tau(b))\}$ to $\{(a, \tau(b)), (b, \tau(a))\}$. Let $\tau' \in MP_n$ be the matching obtained by resolving the crossing this way.

In window notation, we see that

$$
\phi(\tau) = [\ldots, \tau(a) - n, \ldots, \tau(b) - n, \ldots, a + n, \ldots, b + n, \ldots]
$$

and

$$
\phi(\tau') = [\ldots, \tau(b) - n, \ldots, \tau(a) - n, \ldots, b + n, \ldots, a + n, \ldots].
$$
Observe that $\phi(\tau')$ is obtained from $\phi(\tau)$ by swapping the numbers in the $a$-th spot and $b$-th spot and swapping the numbers $a+n$ and $b+n$. Thus, $\phi(\tau') = t_{a+n,b+n}\phi(\tau)t_{a,b}$ where $t_{a,b} = s_as_{a+1} \ldots s_{b-2}s_{b-1}s_b \ldots s_{a+1}s_a$ is the periodic transposition of $a$ and $b$. Therefore, $\phi(\tau')$ is modular palindromic.

Case 2: from $\{(a, \tau(a)), (b, \tau(b))\}$ to $\{(a, b), (\tau(a), \tau(b))\}$.

This case can be proved in exactly the same way as the previous case using transpositions $t_{b-n,\tau(a)-n}$ and $t_{a,\tau(a)}$.

Secondly, we prove that there is no reduced word for $\phi(\tau)$ which contains a subword of $n$ consecutive integers. For the sake of contradiction, suppose there is a reduced word for $\phi(\tau)$ containing a subword of $n$ consecutive integers. Without loss of generality, assume the consecutive integers are increasing. Take a maximal length subword $s_{a+1}s_{a+2} \ldots s_{a+m}$ of $\phi(\tau)$ where $m \geq n$ and the indices are taken modulo $2n$ if necessary. Then, observe that $\phi(\tau)(a+m+1) = a+1$, which contradicts that $i - n < \phi(\tau)(i) < i + n$ by the definition of $\phi$ for all $i$. \hfill \Box

**Corollary 12.** The map $\phi$ is order-reversing. In other words, $\tau' \leq \tau$ in $P_n$ implies $\phi(\tau') \geq \phi(\tau)$ in $\text{MP}_n$ or in $\tilde{S}_{2n}$.

**Proof.** By the proof of the previous lemma, we see that if a matching $\tau'$ is obtained by resolving a crossing of a matching $\tau$, then $\phi(\tau') = t_{a,b}\phi(\tau)t_{a+n,b+n}$ for some $a, b \in [2n]$. Since $\phi(\tau)$ is a subword of $\phi(\tau')$, we conclude that $\phi(\tau) \leq \phi(\tau')$ as required. \hfill \Box

The following lemma will be used when we prove Theorem 10; in particular, we will use the lemma to show that the map $\phi$ is surjective.

**Lemma 13.** Let $p, q \in \text{MP}_n$ such that $q = s_a ps_{a+n}$ and $\ell(q) = \ell(p) + 2$. Let $x, y \in [2n]$ such that $p(x) \equiv a \pmod{2n}$ and $p(y) \equiv a + 1 \pmod{2n}$. Then,

1. The numbers $x, y, a + n$ and $a + n + 1$ are distinct.
2. $q(i) = p(i)$ for all $i \in [2n] \setminus \{a + n, a + n + 1, x, y\}$.
3. $q(a + n) = p(a + n + 1), q(a + n + 1) = p(a + n), q(x) \equiv a + 1 \pmod{2n}$, and $q(y) \equiv a \pmod{2n}$. 

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Proof. (1) It is clear that \( x \neq y \) and \( a + n \neq a + n + 1 \). Since \( p \in \mathcal{MP}_n \), we know \( a < p(a + n) < a + 2n \) and \( a + 1 < p(a + n + 1) < a + 2n + 1 \), and thus \( a + n \neq x \) and \( a + n + 1 \neq y \). Assume \( y = a + n \), then \( p(a + n) \equiv a + 1 \pmod{2n} \). Then \( p \) should contain a subword of the form \( s_{a+1}s_{a+2} \ldots s_{a+n-2}s_{a+n-1} \). Then \( q = s_ap_{s_{a+n}} \) should contain a subword of \( n \) consecutive integers, which contradicts \( q \in \mathcal{MP}_n \). Assume \( x = a + n + 1 \), then \( p(a + n + 1) \equiv a \pmod{2n} \). Then \( p \) should contain a subword of the form \( s_{a+2n+1}s_{a+2n} \ldots s_{a+n+2}s_{a+n+1} \). Then \( q = s_ap_{s_{a+n}} \) should contain a subword of \( n \) consecutive integers, which contradicts \( q \in \mathcal{MP}_n \).

(2) Let \( i \in [2n] \setminus \{a + n, a + n + 1, x, y\} \). Then it is clear that \( s_{a+n}(i) = i \). Note that \( p(i) \notin \{a, a + 1\} \) since \( i \notin \{x, y\} \), and thus \( s_a(p(i)) = p(i) \). Then we have

\[
q(i) = s_ap_{s_{a+n}}(i) = s_a(p(i)) = p(i).
\]

(3) Since \( s_{a+n} \) acts on the right of \( p \) by swapping numbers in the \( (a + n) \)-th spot and \( (a + n + 1) \)-th spot and \( s_a \) acts on the left of \( p \) by swapping numbers \( a \) and \( a + 1 \), we have the second statement. \( \square \)

Proof of Theorem 10. By Lemma 11 and Corollary 12, we need to show that the map \( \phi \) is surjective to complete the proof. Let \( p \in \mathcal{MP}_n \). By definition, no reduced decomposition of \( p \) contains a subword of \( n \) consecutive integers, and thus \( i - n < p(i) < i + n \) for all \( i \in [2n] \). Define \( \tau_p \) by

\[
\tau_p(i) = \begin{cases} 
 p(i) + n & \text{if } p(i) \leq n \\
 p(i) - n & \text{if } p(i) > n + 1 
\end{cases}
\]

for \( i \in [2n] \). We claim that \( \tau_p \) is a matching on \([2n]\). To show \( \tau_p \) is a matching, we need to show that \( \tau_p \) is an involution with no fixed point. In other words, we must show that \( \tau_p(i) \neq i \) and \( \tau_p^2(i) = i \) for all \( i \in [2n] \). Since \( i - n < p(i) < i + n \), either \( i < \tau_p(i) < i + 2n \) or \( i - 2n < \tau_p(i) < i \), and thus \( \tau_p(i) \neq i \). Observe that

\[
\tau_p^2(i) = \begin{cases} 
 p(p(i) + n) + n & \text{if } p(i) \leq n \text{ and } p(p(i) + n) \leq n \\
 p(p(i) + n) - n & \text{if } p(i) \leq n \text{ and } p(p(i) + n) > n + 1 \\
 p(p(i) - n) + n & \text{if } p(i) \geq n + 1 \text{ and } p(p(i) - n) \leq n \\
 p(p(i) - n) - n & \text{if } p(i) \geq n + 1 \text{ and } p(p(i) - n) > n + 1 
\end{cases}
\] (2)

Since \( p \) is an affine permutation in \( \tilde{S}_{2n} \), we see that

\[
p(p(i) - n) + n = p(p(i) + n - 2n) + n = p(p(i) + n) - n
\]

and

\[
p(p(i) - n) - n = p(p(i) + n - 2n) - n = p(p(i) + n) - 3n.
\]

By this observation, we have \( \tau_p^2(i) \equiv p(p(i) + n) + n \pmod{2n} \). Since \( 1 \leq \tau_p(i) \leq 2n \), we can simplify (2) as follows.

\[
\tau_p^2(i) \equiv p(p(i) + n) + n \pmod{2n} \text{ and } 1 \leq \tau_p^2(i) \leq 2n. \] (3)
Now we prove that \( \tau_p \) is an involution by induction on \( \ell(p)/2 \).

Base case: \( p = e \). Observe that

\[
e(e(i) + n) + n = e(i + n) + n = (i + n) + n = i + 2n,
\]

and \( \tau^2_p(i) = i \) for all \( i \in [2n] \) as desired.

Induction step: Suppose \( \tau^2_p(i) = i \) for all \( i \in [2n] \) for \( p \in \mathcal{MP}_n \). Then \( p(p(i) + n) + n \equiv i \mod 2n \), or equivalently, \( p(p(i) + n) \equiv i + n \mod 2n \). Let \( q = s_a p s_{a+n} \in \mathcal{MP}_n \) with \( \ell(q) = \ell(p) + 2 \). Let \( x, y \in [2n] \) such that \( p(x) \equiv a \mod 2n \) and \( p(y) \equiv a + 1 \mod 2n \). Let \( i \in [2n] \setminus \{a + n, a + n + 1, x, y\} \). By Lemma 13, we see \( p(i) + n \not\equiv a + n \mod 2n \) and \( p(i) + n \not\equiv a + n + 1 \mod 2n \), and thus

\[
q(q(i) + n) + n \equiv q(p(i) + n) + n \\
\equiv (s_a p s_{a+n})(p(i) + n) + n \\
\equiv s_a(p(i) + n) + n \\
\equiv s_a(i + n) + n \\
\equiv i + 2n,
\]

where the last equality is due to \( i \not\in \{a + n, a + n + 1\} \), and hence \( \tau^2_q(i) = i \) for \( i \in [2n] \setminus \{a + n, a + n + 1, x, y\} \). We calculate \( q(q(i) + n) + n \) for \( i \in \{a + n, a + n + 1, x, y\} \) as follows.

\[
q(q(a + n) + n) + n \equiv a + 3n, \\
q(q(a + n + 1) + n) + n \equiv a + 3n + 1, \\
q(q(x) + n) + n \equiv x + 2n, \\
q(q(y) + n) + n \equiv y + 2n,
\]

and thus \( \tau^2_q(i) = i \) for \( i \in \{a + n, a + n + 1, x, y\} \). Hence by induction, \( \tau_p \) is an involution and therefore \( \tau_p \) is a matching on \([2n]\). By the construction of \( \tau_p \), observe that

\[
\phi(\tau_p)(i) = h_{\tau_p}(i) = \begin{cases} 
\tau_p(i) - n = (p(i) + n) - n & \text{if } i < \tau_p(i) \Leftrightarrow p(i) \leq n \\
\tau_p(i) + n = (p(i) - n) + n & \text{if } i > \tau_p(i) \Leftrightarrow p(i) \geq n + 1
\end{cases}
\]

and thus \( \phi(\tau_p)(i) = p(i) \) for all \( i \in [2n] \), which shows that the map \( \phi \) is surjective. \( \square \)

5 The cd-index of the posets \( P_n \)

In the previous section, we saw that the poset \( P_n \), without \( \hat{0} \) adjoined, is isomorphic to a subposet of the dual Bruhat order of affine permutations. Reading [13] provided a recursive formula for the cd-indices of intervals in the Bruhat order on a Coxeter group, and it looks promising to examine the recursion to compute the cd-index of \( P_n \).

The product of a poset \( P \) with a chain of length one is called the \textit{pyramid} of \( P \), and denoted by \( \text{Pyr}(P) \). We will use the proposition from Ehrenborg-Readdy [7] which produces the cd-index of \( \text{Pyr}(P) \) from the cd-index of \( P \).
Proposition 14 ([7], Proposition 4.2). Let $P$ be an Eulerian poset. Then

$$\Phi_{P_{yr}(P)} = \frac{1}{2} \left( \Phi_p \cdot c + c \cdot \Phi_p + \sum_{0 < x < 1} \Phi_{[0,x]} \cdot d \cdot \Phi_{[x,1]} \right).$$

A zipper in a poset $P$ is a triple of distinct elements $x, y, z \in P$ such that $\{w : w < x\} = \{w : w < y\}$ and $z = x \lor y$ covers $x$ and $y$ but covers no other element. Please see Figure 2.

![Figure 2: A zipper $\{x, y, z\}$ and zip of the zipper](image)

The operation zip of a zipper $\{x, y, z\} \subset P$ is defined as follows. Let $xy$ be a single new element not in $P$, and define $P' = (P - \{x, y, z\}) \cup \{xy\}$, with a binary relation $\preceq$ on $P'$, given by:

- $a \preceq b$ if $a \preceq b$ in $P - \{x, y, z\}$
- $xy \preceq a$ if $x \preceq a$ or if $y \preceq a$ in $P - \{x, y, z\}$
- $a \preceq xy$ if $a \preceq x$ or (equivalently) if $a \preceq y$ in $P - \{x, y, z\}$
- $xy \preceq xy$.

We can think of the operation zip of a zipper $\{x, y, z\}$ as deleting $z$ and identifying $x$ with $y$. The zip operation of a zipper produces a new poset. The zip operation of an Eulerian poset is also Eulerian. Furthermore, the following theorem provides a formula for the cd-index of the resulting poset in terms of the cd-index of the initial Eulerian poset.

Theorem 15 ([13], Theorem 4.6). $P'$ is a poset under the partial order $\preceq$. If $P$ is Eulerian then so is $P'$. Moreover, $P$ has a cd-index $\Phi_p$ if and only if $P'$ has a cd-index $\Phi_{p'}$, and

$$\Phi_{p'} = \Phi_p - \Phi_{[0,x]} \cdot d \cdot \Phi_{[x,1]}.$$
The following theorem is a structural recursion for Bruhat intervals.

**Theorem 16** ([13], Theorem 5.5). Let \((W, S)\) be a Coxeter system. Let \(w, u \in W\) and \(s \in S\). Let \(ws > w, us > u\) and \(u \leq w\).

1. If \(us \notin [u, w]\) then \([u, ws] \cong [u, w] \times [1, s]\) and \([us, ws] \cong [u, w]\).

2. If \(us \in [u, w]\), then \([u, ws]\) can be obtained from \([u, w] \times [1, s]\) by a sequence of zippings.

From these two theorems, Reading [13] produced recursions for the cd-indices of Bruhat intervals. For \(v \in W\) and \(s \in S\), define \(\sigma_s(v) := \ell(vs) − \ell(v) \in \{-1, 1\}\).

**Theorem 17** ([13], Theorem 6.1). Let \((W, S)\) be a Coxeter system. Let \(w, u \in W\) and \(s \in S\). Let \(u < us, w < ws\) and \(u \leq w\).

1. If \(us \notin [u, w]\), then \(\Phi[us, ws] = \Phi[P_{yr}([u, w])] = \frac{1}{2} \left( \Phi[u, w] \cdot c + c \cdot \Phi[u, w] + \sum_{v : u < v < w} \Phi[u, v] \cdot d \cdot \Phi[v, w] \right)\).

2. If \(us \in [u, w]\), then \(\Phi[u, ws] = \Phi[P_{yr}([u, w])] - \sum_{v : u < v < w} \Phi[u, v] \cdot d \cdot \Phi[v, w]\)
   \[= \frac{1}{2} \left( \Phi[u, w] \cdot c + c \cdot \Phi[u, w] + \sum_{v : u < v < w} \sigma_s(v) \Phi[u, v] \cdot d \cdot \Phi[v, w] \right) \]

We claim that there are recurrence relations of the cd-indices of intervals in \(MP_n\), and also in \(P_n\). For \(v \in MP_n\) and \(s \in S\), define \(\sigma_s(v) := \frac{1}{2} (\ell(sv) − \ell(v)) \in \{-1, 1\}\).

**Theorem 18.** Let \(u, w \in MP_n\) and let \(s = s_i\) and \(s' = s_{i+n}\) for some \(0 \leq i < 2n\). Let \(u < sus', w < sws'\) and \(u \leq w\).

1. If \(sus' \notin [u, w]\), then \(\Phi[sus', sws'] = \Phi[u, w]\), and
   \[\Phi[u, sus'] = \Phi[P_{yr}([u, w])] = \frac{1}{2} \left( \Phi[u, w] \cdot c + c \cdot \Phi[u, w] + \sum_{v : u < v < w} \Phi[u, v] \cdot d \cdot \Phi[v, w] \right) \]
(2) If $sus' \in [u, w]$, then

$$
\Phi_{[u, sus']} = \Phi_{pyr([u, w])} - \sum_{v \in MP_2 \atop u < v < w} \Phi_{[u, v]} \cdot d \cdot \Phi_{[v, w]}
$$

$$
= \frac{1}{2} \left( \Phi_{[u, w]} \cdot c + c \cdot \Phi_{[u, u]} + \sum_{v \in MP_2 \atop u < v < w} \sigma(v) \Phi_{[u, v]} \cdot d \cdot \Phi_{[v, w]} \right).
$$

Example 19. Let $u = e$ and $w = s_1 s_2 s_5 s_4$ in $MP_3$. The Hasse diagrams for intervals $[u, w], [s_3 s_0, s_3 s_0], [u, s_3 s_0]$ and $[u, s_2 s_5]$ are in Figure 3.

![Figure 3: The Hasse diagrams for some intervals in MP_3](image)

(1) Let $s = s_3$ and $s' = s_0$. Observe that $u < sus', w < sws', u \leq w$ and $sus' \notin [u, w]$. We see that $\Phi_{[sus', sws']} = \Phi_{[s_3 s_0, s_3 s_1 s_2 s_5 s_4 s_0]} = c = \Phi_{[e, s_3 s_1 s_2 s_5 s_4]} = \Phi_{[u, w]}$. Notice that the intervals $[u, w]$ and $[s_3 s_0, s_3 s_0]$ are isomorphic to Boolean lattice $B_2$. The
The cd-index of $B_2$ is $\Phi_{B_2}(c, d) = c$. We also have

\[
\Phi_{[u, ws]} = \Phi_{[e, s_3 s_1 s_2 s_5 s_4 s_0]}
\]

\[
= \frac{1}{2} \left( \Phi_{[e, s_1 s_2 s_5 s_4]} \cdot c + c \cdot \Phi_{[e, s_1 s_2 s_5 s_4]} + \sum_{c < v < s_1 s_2 s_5 s_4} \Phi_{[e, v]} \cdot d \cdot \Phi_{[v, s_1 s_2 s_5 s_4]} \right)
\]

\[
= \frac{1}{2} \left( c \cdot c + c \cdot c + \Phi_{[e, s_1 s_4]} \cdot d \cdot \Phi_{[s_1 s_4, s_1 s_2 s_5 s_4]} + \Phi_{[e, s_2 s_5]} \cdot d \cdot \Phi_{[s_2 s_5, s_1 s_2 s_5 s_4]} \right)
\]

\[
= \frac{1}{2} (c^2 + c^2 + d + d) = c^2 + d.
\]

Note that the interval $[u, s_3 ws_0]$ is isomorphic to Boolean lattice $B_3$. The cd-index of $B_3$ is $\Phi_{B_3}(c, d) = c^2 + d$.

(2) Let $s = s_2$ and $s' = s_5$. Observe that $u < sus'$, $w < ws', u \leq w$ and $us' \in [u, w]$. Note that $\sigma_{s_2}(s_1 s_4) = \frac{1}{2} (4 - 2) = 1$ and $\sigma_{s_2}(s_2 s_5) = \frac{1}{2} (0 - 2) = -1$. We see that

\[
\Phi_{[u, ws']} = \Phi_{[e, s_2 s_1 s_2 s_5 s_4 s_5]}
\]

\[
= \frac{1}{2} \left( \Phi_{[e, s_1 s_2 s_5 s_4]} \cdot c + c \cdot \Phi_{[e, s_1 s_2 s_5 s_4]} + \sum_{c < v < w} \sigma_{s_2} (v) \Phi_{[e, v]} \cdot d \cdot \Phi_{[v, s_1 s_2 s_5 s_4]} \right)
\]

\[
= \frac{1}{2} \left( c \cdot c + c \cdot c + \Phi_{[e, s_1 s_4]} \cdot d \cdot \Phi_{[s_1 s_4, s_1 s_2 s_5 s_4]} - \Phi_{[e, s_2 s_5]} \cdot d \cdot \Phi_{[s_2 s_5, s_1 s_2 s_5 s_4]} \right)
\]

\[
= \frac{1}{2} (c^2 + c^2 + d - d) = c^2.
\]

Observe that the interval $[u, s_2 ws_5]$ is isomorphic to Bruhat order of symmetric group $S_3$. The cd-index of Bruhat order of $S_3$ is $\Phi_{S_3}(c, d) = c^2$.

6 Proof of Theorem 18

In this section we let $u, w \in \mathcal{MP}_n$ and let $s = s_i \in \tilde{S}$ and let $s' = s_{i+n} \in \tilde{S}$. Suppose $u < sus'$ and $w < ws'$. We slightly modify Reading’s [13] map $\eta$ to define a map $\eta : [u, w] \times [e, ss'] \rightarrow [u, ws']$, as follows:

\[
\eta(v, e) = v
\]

\[
\eta(v, ss') = \begin{cases} 
  sus' & \text{if } sus' > v \\
  v & \text{if } sus' < v
\end{cases}
\]

The following proposition is known as Lifting Property of Bruhat order.

Proposition 20 ([4], Proposition 1.2 (Lifting Property of Bruhat order)). Let $(W, S)$ be a Coxeter system. Let $u, w \in W$ and $s \in S$. If $w > ws$ and $us > u$, then the following are equivalent:

(i) $w > u$
(ii) \( ws > u \)

(iii) \( w > us \).

We will need the following proposition, which is the analogue of Proposition 20 for \( \mathcal{MP}_n \).

**Proposition 21** (Lifting Property of \( \mathcal{MP}_n \)). Let \( u, w \in \mathcal{MP}_n \) and let \( s = s_i \in \tilde{S} \). Let \( s' = s_{i+n} \in \tilde{S} \). If \( w > sws' \) and \( sus' > u \), then the following are equivalent:

(i) \( w > u \)

(ii) \( sws' > u \)

(iii) \( w > sus' \).

**Proof.** Since \( w > sws' \), by transitivity (ii) implies (i). Since \( sus' > u \), by transitivity (iii) implies (i). So assume (i) \( w > u \). Choose a reduced decomposition for \( sws' = t_1t_2\ldots t_q \) where \( t_i \in \tilde{S} \) for all \( i \in [2q] \) such that the reduced word is modular palindromic. Then \( w = st_1t_2\ldots t_q s' \) is also reduced and modular palindromic. There is a reduced decomposition for \( u \)

\[
u = t_1t_2\ldots t_{2r},
\]

which is a subword of \( w = st_1t_2\ldots t_q s' \). Since \( sus' > u \), we have \( t_{i_1} \neq s \) and \( t_{i_2r} \neq s' \), and thus both (ii) \( sws' > u \) and (iii) \( w > sus' \) hold as desired.

We claim that the map \( \eta \) is well-defined. To show this, let \( v \in [u, w] \). Because we have assumed \( w < sws' \) we know that \( v \in [u, w] \subset [u, sws'] \), so we may assume that \( \eta(v, ss') = sws' \). In this case, \( v < sws' \), thus we see \( u \leq v < sws' < sws' \) where the last inequality is due to the lifting property, and therefore \( \eta(v, ss') \in [u, sws'] \). The following proposition and Proposition 5.1 in [13] are the same statement on different posets: Bruhat order of Coxeter groups and the induced subposet \( \mathcal{MP}_n \). Here, we show the order-preserving part of the statement on the induced subposet \( \mathcal{MP}_n \). For the proof of surjective part, see Proposition 5.1 in [13].

**Proposition 22.** If \( u < sus' \) and \( w < sws' \), then \( \eta : [u, w] \times [e, ss'] \rightarrow [u, sws'] \) is an surjective order-preserving map.

**Proof.** Suppose \( (v_1, a_1) \leq (v_2, a_2) \) in \([u, w] \times [e, ss']\). Since \( e \leq a_1 \leq a_2 \leq ss' \), so we break up into three cases.

Case 1: \( a_1 = e \) and \( a_2 = e \). Then \( \eta(v_1, a_1) = v_1 \leq v_2 = \eta(v_2, a_2) \).

Case 2: \( a_1 = e \) and \( a_2 = ss' \). Then either \( \eta(v_2, a_2) = v_2 \) with \( v_2 > sv_2s' \) or \( \eta(v_2, a_2) = sv_2s' \) with \( sv_2s' > v_2 \). In either cases, we see that \( \eta(v_2, a_2) \geq v_1 \).

Case 3: \( a_1 = ss' \) and \( a_2 = ss' \). Then either \( \eta(v_1, a_1) = v_1 \) with \( v_1 > sv_1s' \) or \( \eta(v_1, a_1) = sv_1s' \) with \( sv_1s' > v_1 \). We also have either \( \eta(v_2, a_2) = v_2 \) with \( v_2 > sv_2s' \) or \( \eta(v_2, a_2) = sv_2s' \) with \( sv_2s' > v_2 \). We break up this case into four subcases.
Subcase 3-1: \( \eta(v_1, a_1) = v_1 \) and \( \eta(v_2, a_2) = v_2 \). Then, \( \eta(v_1, a_1) \leq \eta(v_2, a_2) \).

Subcase 3-2: \( \eta(v_1, a_1) = v_1 \) and \( \eta(v_2, a_2) = sv_2 s' \). Then, \( \eta(v_1, a_1) = v_1 \leq v_2 < sv_2 s' = \eta(v_2, a_2) \).

Subcase 3-3: \( \eta(v_1, a_1) = sv_1 s' \) and \( \eta(v_2, a_2) = v_2 \). Then, \( \eta(v_1, a_1) = sv_1 s' \leq v_2 = \eta(v_2, a_2) \) by the lifting property.

Subcase 3-4: \( \eta(v_1, a_1) = sv_1 s' \) and \( \eta(v_2, a_2) = sv_2 s' \). Then, \( \eta(v_1, a_1) = sv_1 s' \leq sv_2 s' = \eta(v_2, a_2) \) by the lifting property.

For every \( v \in [u, w] \) with \( sv_s s' < v \), notice that the image of the elements \((v, e), (sv_{s'} s, ss')\) under the map \( \eta \) is the single element \( v \). From this observation, we let \( v_1, v_2, \ldots, v_k \) be a linear ordering of the elements of the set \( Z = \{ v : u < v < w, sv_s s' < v \} \) such that the ranks (or lengths) of elements are weakly increasing, in other words, \( \ell(v_1) \leq \ell(v_2) \leq \ldots \leq \ell(v_k) \). Define posets \( Q_i \) for \( 0 \leq i \leq k \) recursively as follows. Let \( Q_0 = [u, w] \times [e, ss'] \). Let \( Q_1 \) to be the poset obtained by zipping \( \{ (v_1, e), (sv_1 s', ss'), (v_1, ss') \} \) in \( Q_{i-1} \). In the following proposition, we show that this is indeed a proper zipping.

**Proposition 23.** The triples \( \{(v_1, e), (sv_1 s', ss'), (v_1, ss')\} \) in \( Q_{i-1} \) for \( 1 \leq i \leq k \) are zippers.

**Proof.** First, we claim that \( (v_1, e), (sv_1 s', ss') \) and \( (v_1, ss') \) are elements of \( Q_{i-1} \). The element \( (v_1, ss') \) has not been deleted yet, and we have not identified \( (v_1, ss') \) with any element because it is at a rank higher than we have yet made identifications. The only elements ever deleted are of the form \( (x, ss') \) where \( x > ss' \), so \( (v_1, e) \) and \( (sv_1 s', ss') \) have not been deleted. The only identification one could make involving \( (v_1, e) \) and \( (sv_1 s', ss') \) is to identify them to each other, and that has not happened yet, and thus the claim is proved. Second, we check the conditions in the definition of a zipper. Suppose \( (x, a) < (v_1, e) \), then \( x < v_1 \) and \( a = e \). Then \( x < ss' \) otherwise the triple \( \{(sx s', e), (x, ss'), (x, e)\} \) is a zipper in \( Q_{i-1} \) with \( \ell(x) < \ell(v_1) \) which is impossible. Then we have \( x \leq sv_1 s' \) by lifting property and thus \( (x, a) < (sv_1 s', ss') \). Now suppose \( (x, a) < (sv_1 s', ss') \), then either \( a = e \) or \( a = ss' \). If \( a = e \) then \( (x, a) < (v_1, e) \) since \( x \leq sv_1 s' < v_1 \). Assume \( a = ss' \). Then \( x \leq sv_1 s' \) which implies \( \ell(x) < \ell(sv_1 s') \) so \( x \) is not \( v_1 \) for any \( r \leq i \), and then \( x < ss' \). Then the triple \( \{ (x, ss'), (sx s', e), (sx s', ss') \} \) is a zipper with \( \ell(ss') < \ell(v_1) \) in \( Q_{i-1} \) which is impossible. Hence the triple satisfies the first condition.
\{(x, a) : (x, a) < (v_i, e)\} = \{(x, a) : (x, a) < (sv_i s', ss')\}. The second condition is obvious because \((v_i, e) \leq (v_i, ss')\) and \((sv_i s', ss') \leq (v_i, ss')\). Assume \((x, a)\) is covered by \((v_i, ss')\). If \(a = e\) then \(x = v_i\), so \((x, a) = (v_i, e)\). If \(a = ss'\) then \(x < v_i\). Since \(sv_i s' < v_i\) and \(x < sx s'\), by lifting property \(x \leq sv_i s'\) which implies \(x = sv_i s'\). Hence \((v_i, ss')\) covers no other element than \((v_i, e)\) and \((sv_i s', ss')\), and thus the third condition holds. Therefore, the triple \\{\((v_i, e), (sv_i s', ss'), (v_i, ss')\)\} is a zipper in \(Q_i - 1\).

In terms of zipper and operation zip, we can think of the image of \([u, w] \times [e, ss']\) under the map \(\eta\) as a sequence of zipping operations of the triples \\{\((v_i, e), (sv_i s', ss'), (v_i, ss')\)\} for \(v_i \in Z\).

**Example 24.** Let \(u = e, w = s_1 s_3 s_2 s_6 s_7 s_5\) in \(MP_4\) and let \(s = s_2\) and \(s' = s_6\). Then, \(u < su s', w < su s'\) and \(u < w\). The Hasse diagram of \([u, w] \times [e, ss']\) is

![Hasse diagram](image)

We consider the map \(\eta : [u, w] \times [e, ss'] \to [u, su s']\). After zipping the zipper \\{\((s_2 s_6, e)\), \((e, s_2 s_6)\), \((s_2 s_6, s_2 s_6)\)\}, we get

We consider the map \(\eta : [u, w] \times [e, ss'] \to [u, su s']\). After zipping the zipper \\{\((s_2 s_6, e)\), \((e, s_2 s_6)\), \((s_2 s_6, s_2 s_6)\)\}, we get

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The following proposition is an analogue of Proposition 5.2 in [13]. We omit the proof of the following proposition, and instead refer to Proposition 5.2 in [13].

**Proposition 25.** Let \( u < sus' \) and \( w < sws' \) and \( sus' \not\leq w \). Then \( svv' > v \) for all \( v \in [u, w] \), and \( \eta \) is an order-preserving bijection.

Proposition 25 directly implies the following corollary.

**Corollary 26.** Let \( u < sus' \) and \( w < sws' \) and \( sus' \not\leq w \). Then the map \( \theta : [u, w] \to [sus', sws'] \) with \( \theta(v) = svv' \) is an order-preserving bijection.

In other words, if the condition in Corollary 26 holds, then the map \( \theta \) makes a copy \([sus', sws']\) of the interval \([u, w]\).

We have proven the following theorem which is an analogue of Theorem 5.5 in [13].

**Theorem 27.** Let \( w < sws' \) and \( u < sus' \) and \( u \leq w \). If \( sus' \not\in [u, w] \) then \([u, sws'] \cong [u, w] \times [1, ss']\) and \([sus', sws'] \cong [u, w] \). If \( sus' \in [u, w] \), then \([u, sws']\) can be obtained from \([u, w] \times [1, ss']\) by a sequence of zippings.

Now we are ready to prove Theorem 18. This proof is essentially the same as the proof of Theorem 6.1 in [13]. Here we reproduce this proof for completeness.

**Proof of Theorem 18.** Part (1) follows from Proposition 25 and Corollary 26. So we focus on part (2), in the case of \( sus' \in [u, w] \). Define the posets \( Q_i \) as in Proposition 23. By Theorem 15,

\[
\Phi_{Q_{i-1}} - \Phi_{Q_i} = \Phi_{[(u, e), (v_i, e)]} \cdot d \cdot \Phi_{[(v_i, ss'), (u, ss')]}.
\]
where intervals in the right hand side are in \( Q_{j-1} \). Since \( Q_k = [u, ws's'] \), sum from \( i = 1 \) to \( i = k \) to obtain
\[
\Phi_{[u, ws's']} = \Phi_{Q_0} - \sum_{j=1}^{k} \Phi_{[(u,e),(v_j,e)]} \cdot d \cdot \Phi_{[(v_j,ss'),(w,ss')]}.
\]
where intervals in right hand side are in \( Q_{j-1} \). Notice that the interval \([(u,e),(v_j,e)]\) in \( Q_{j-1} \) is isomorphic to the interval \([(u,e),(v_j,e)]\) in \( Q_0 \) which is also isomorphic to \([u,v]\). Similarly, the interval \([(v_j,ss'),(w,ss')]\) in \( Q_{j-1} \) is isomorphic to the interval \([(v_j,ss'),(w,ss')]\) in \( Q_0 \) which is isomorphic to \([v_j,w]\), and the second part of the theorem is proved.

Therefore, we have a recurrence relation for the cd-indices \( \Phi(c,d) \) of intervals in \( P_n \), or equivalently, \( \mathcal{MP}_n \). Then we also have a recurrence relation for the ab-indices \( \Psi(a,b) \) of the intervals by the relation \( \Psi(a,b) = \Phi(a+b,ab+ba) \). Recall that poset \( \hat{P}_n \) is \( P_n \) with a unique minimum element \( \hat{0} \) adjoined. We prove the following proposition which helps us compute ab-indices of intervals \([0, \tau]\) in \( \hat{P}_n \) recursively.

**Proposition 28.** Let \( P \) be a graded poset of rank \( n \) with a unique maximum element \( \hat{1} \) and multiple minimal elements such that every interval in \( P \) is Eulerian. Let \( \hat{P} \) be the poset \( P \) with a unique minimum element \( \hat{0} \) adjoined, where \( \rho(\hat{0}) = -1 \). If \( \hat{P} \) is Eulerian, then the ab-index of \( \hat{P} \) is given by

\[
\Psi_{\hat{P}}(a,b) = (a-b)^n + \sum_{i=0}^{n-1} (a-b)^i b \sum_{\rho(x)=i} \Psi_{[x,\hat{1}]}(a,b).
\]

**Proof.** By the definition of ab-index and the definition of flag \( f \)-vector and \( h \)-vector,

\[
\Psi_{\hat{P}}(a+b,b) = \sum_{S \subseteq [n-1] \cup \{0\}} \alpha_{P}(S)u_S
= \sum_{\hat{0} < t_1 < \cdots < t_{k-1} < \hat{1}} a^{\rho(\hat{0},t_1)-1} b a^{\rho(t_1,t_2)-1} b \cdots b a^{\rho(t_{k-1},\hat{1})-1},
\]

and thus

\[
\Psi_{\hat{P}}(a,b) = \sum_{\hat{0} < t_1 < \cdots < t_{k-1} < \hat{1}} (a-b)^{\rho(\hat{0},t_1)-1} b (a-b)^{\rho(t_1,t_2)-1} b \cdots b (a-b)^{\rho(t_{k-1},\hat{1})-1}.
\]

We rearrange this summation in terms of \( t_1 \), the lowest element in the chain except \( \hat{0} \). If there is no \( t_1 \) in the chain, then the summand will be \((a-b)^n\). If \( \rho(t_1) = i \in \{0, 1, \ldots, n-1\} \), then the summand will be \((a-b)^i b[(a-b)^{\rho(t_1,t_2)-1} b \cdots b (a-b)^{\rho(t_{k-1},\hat{1})-1}]\). By this observation, we can write (4) as

\[
\Psi_{\hat{P}}(a,b) = (a-b)^n + \sum_{i=0}^{n-1} (a-b)^i b \sum_{t_1 < \cdots < t_{k-1} < \hat{1}} (a-b)^{\rho(t_1,t_2)-1} b \cdots b (a-b)^{\rho(t_{k-1},\hat{1})-1}
= (a-b)^n + \sum_{i=0}^{n-1} (a-b)^i b \sum_{\rho(t_1)=i} \Psi_{[t_1,\hat{1}]}(a,b).
\]
as desired.

The previous proposition directly implies the following theorem. We have recursions for the ab-indices of the poset $P_n$ and its intervals $[0, \tau]$.

**Theorem 29.** The ab-index of $\hat{P}_n$ is recursively given by

$$
\Psi_{\hat{P}_n}(a, b) = (a - b)\binom{n}{2} - \sum_{i=0}^{n/2} (a - b)^i b \sum_{\ell(x) = 2\binom{n}{2} - 2i} \Psi_{[e, x]}(a, b)
$$

where $x \in MP_n$. Let $\tau \in P_n$ such that $c(\tau) = k \leq \binom{n}{2}$ and $\phi(\tau) = w \in MP_n$. The ab-index of the interval $[0, \tau] \subset \hat{P}_n$ is recursively given by

$$
\Psi_{[0, \tau]}(a, b) = (a - b)^k + \sum_{i=0}^{k-1} (a - b)^i b \sum_{x: x > w \ell(x) = 2\binom{n}{2} - 2i} \Psi_{[w, x]}(a, b).
$$

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**References**


